Toward a Theory of Meantone (and 31-E.T.) Harmony

diagrams

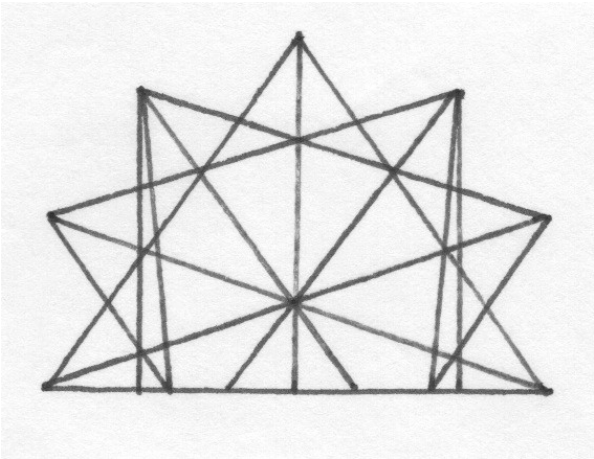

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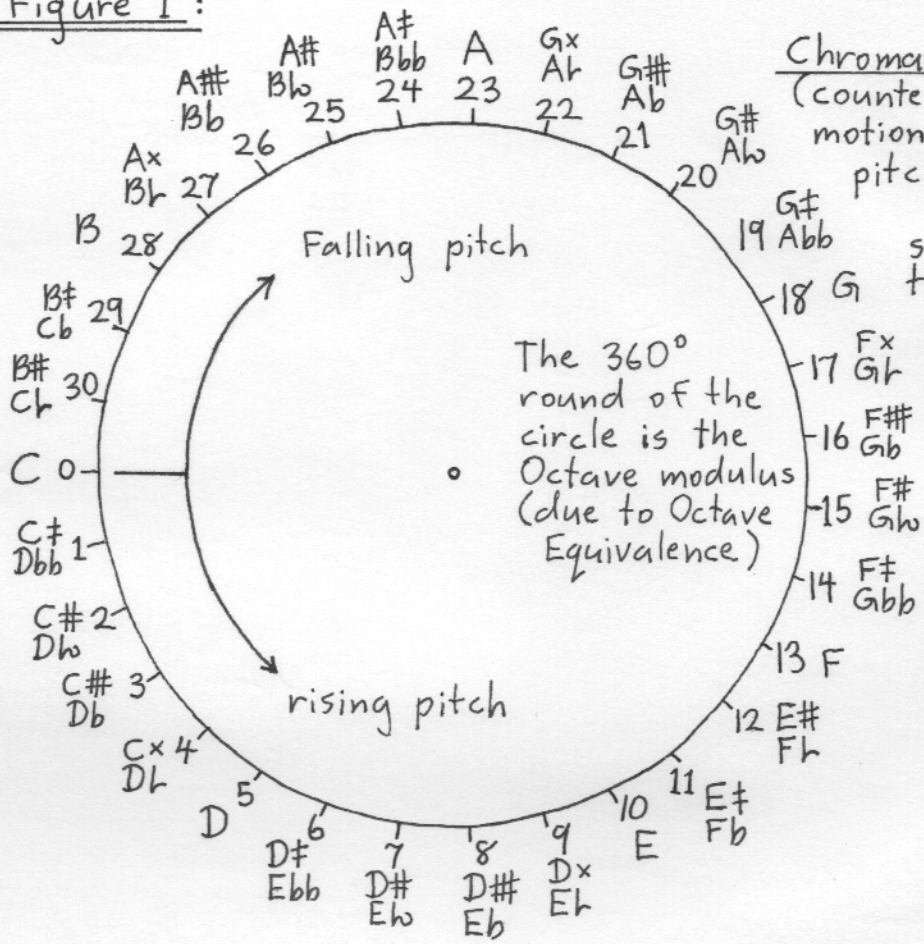
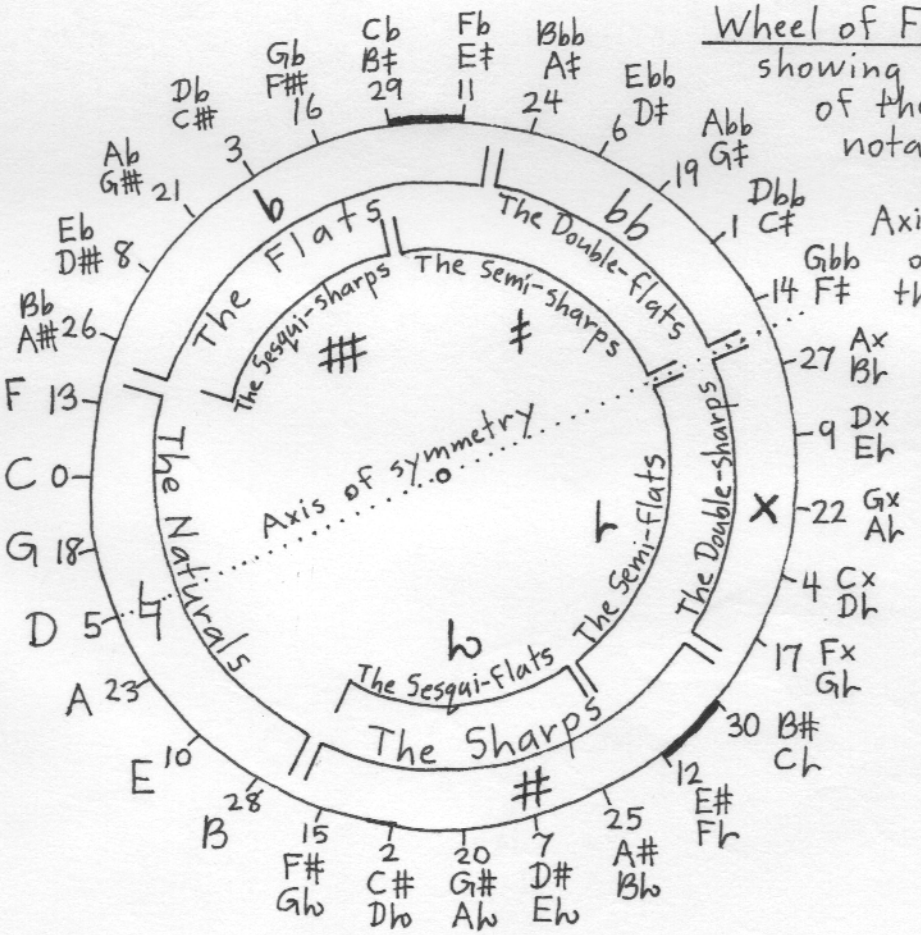
Figure 1:



Chromatic Wheel (counter-clockwise motion is rising pitch) showing both the numerical notation and the Fokker notation

Notation signs
(♮) natural
♯ semi-sharp
sharp
sesqui-sharp
x double-sharp
h semi-Flat
b Flat
bh sesqui-flat
bb double-flat

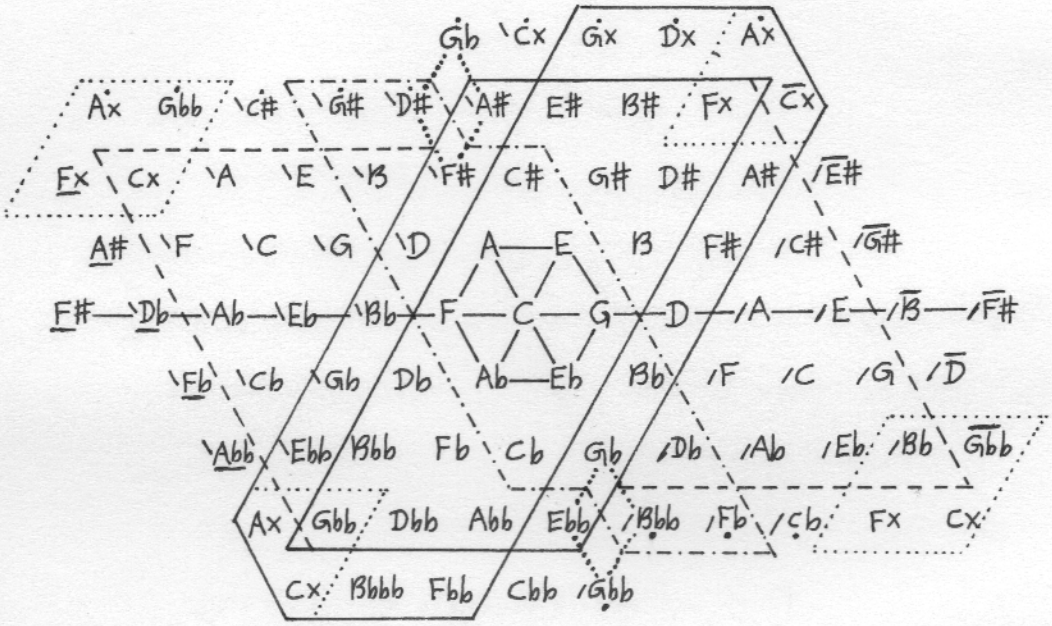


Wheel of Fifths-Fourths showing the logic of the Fokker notation with its Axis of Symmetry on "D" and the "Dorian" Mode.

At the opposite end of the Axis note the "gap" between pitches 14 and 27.

Figure 2:

The Field of Just Intonation showing the comma-based notation:



The Field of Just Intonation showing the prominant regions:

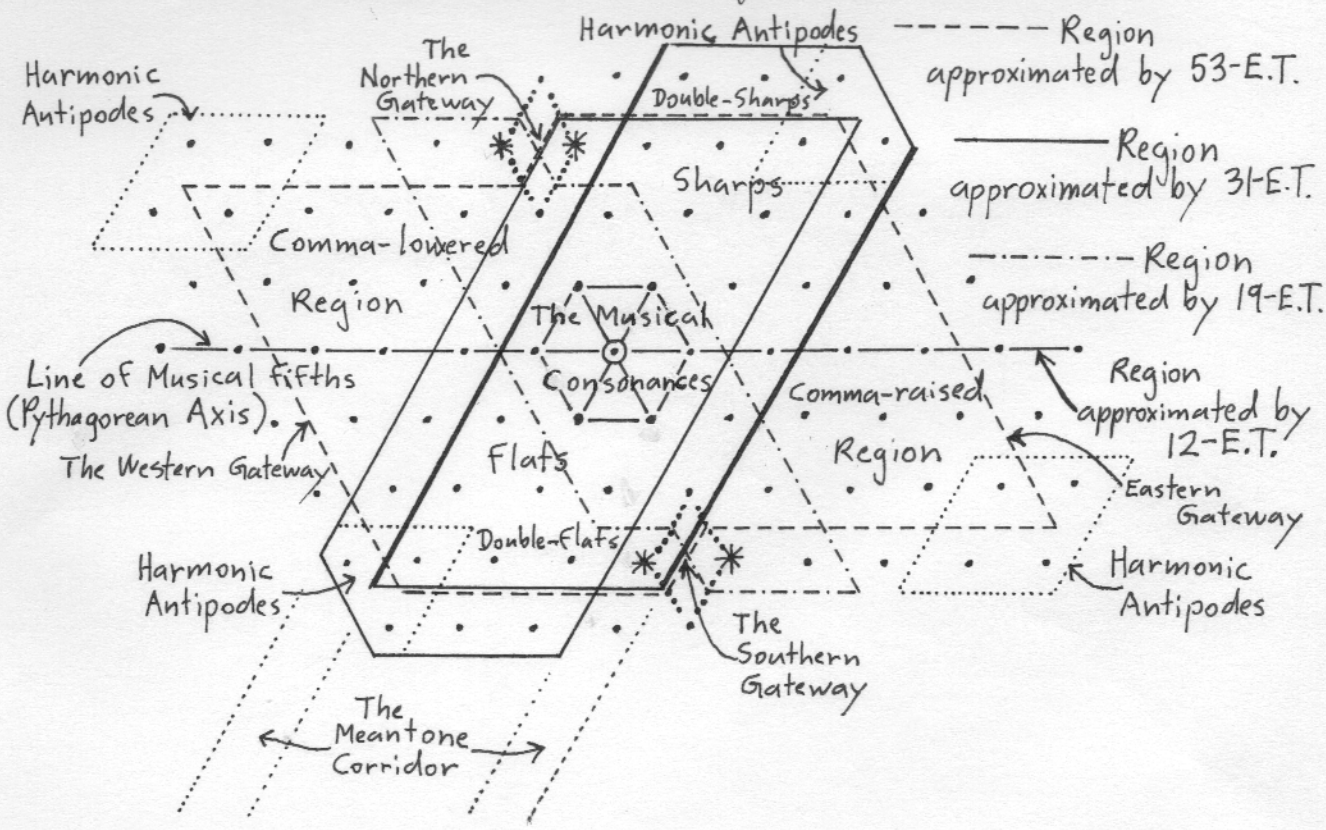
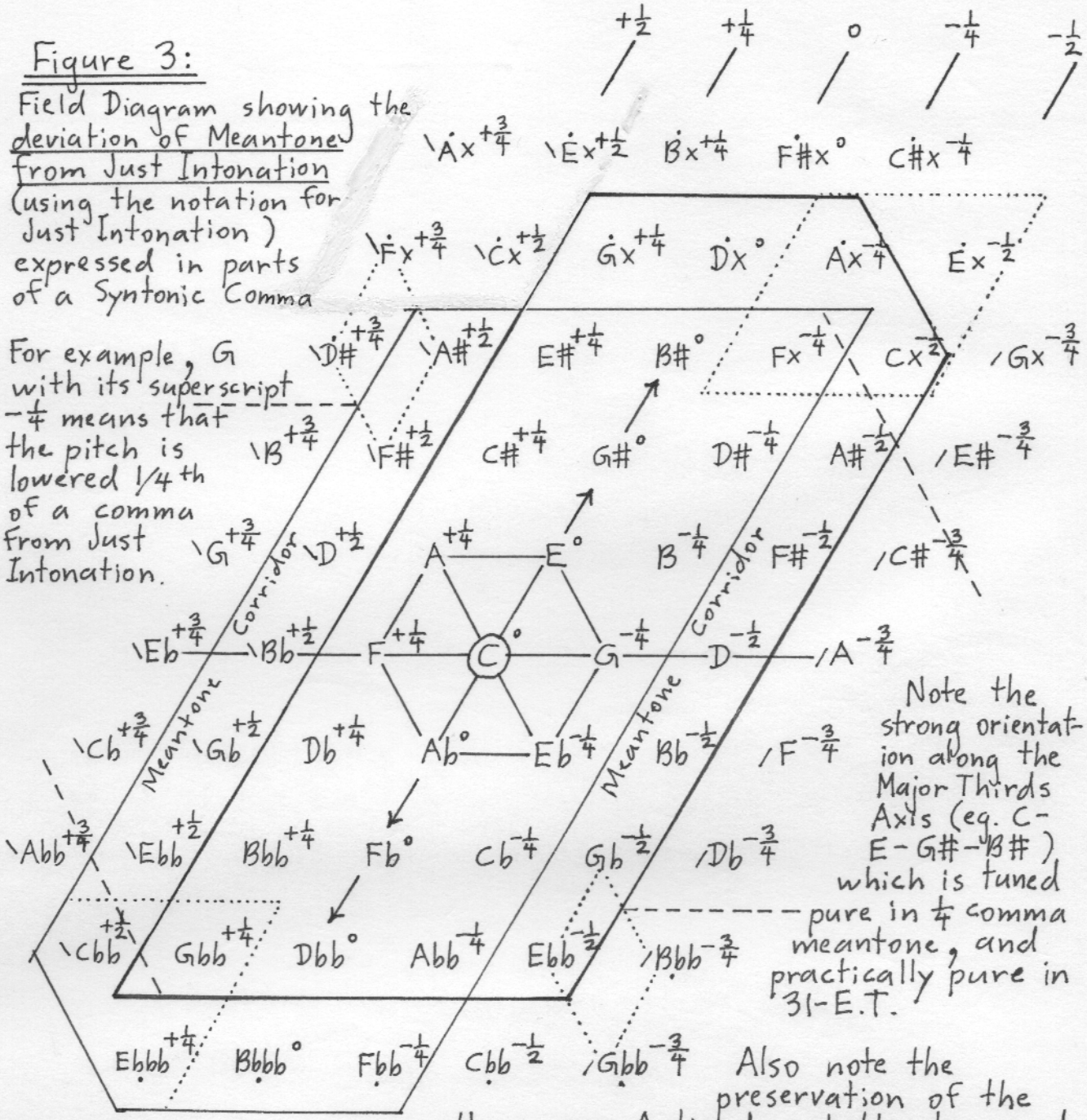


Figure 3:

Field Diagram showing the deviation of Meantone from Just Intonation (using the notation for Just Intonation) expressed in parts of a Syntonic Comma

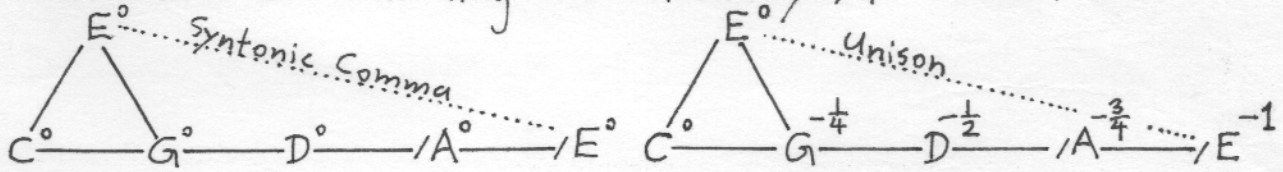
For example, G with its superscript $-\frac{1}{4}$ means that the pitch is lowered 1/4th of a comma from Just Intonation.



Note the strong orientation along the Major Thirds Axis (eg. C-E-G#-B#) which is tuned pure in 1/4 comma meantone, and practically pure in 31-E.T.

Also note the preservation of the Harmonic Antipodes at the top and bottom of the Field. In the closed cycle of 31-E.T., these two ends meet in a "strange loop" at the opposite side of the circle of Fifths from the Generator tone "C."

Rationale for flattening the Fifths by 1/4th comma:

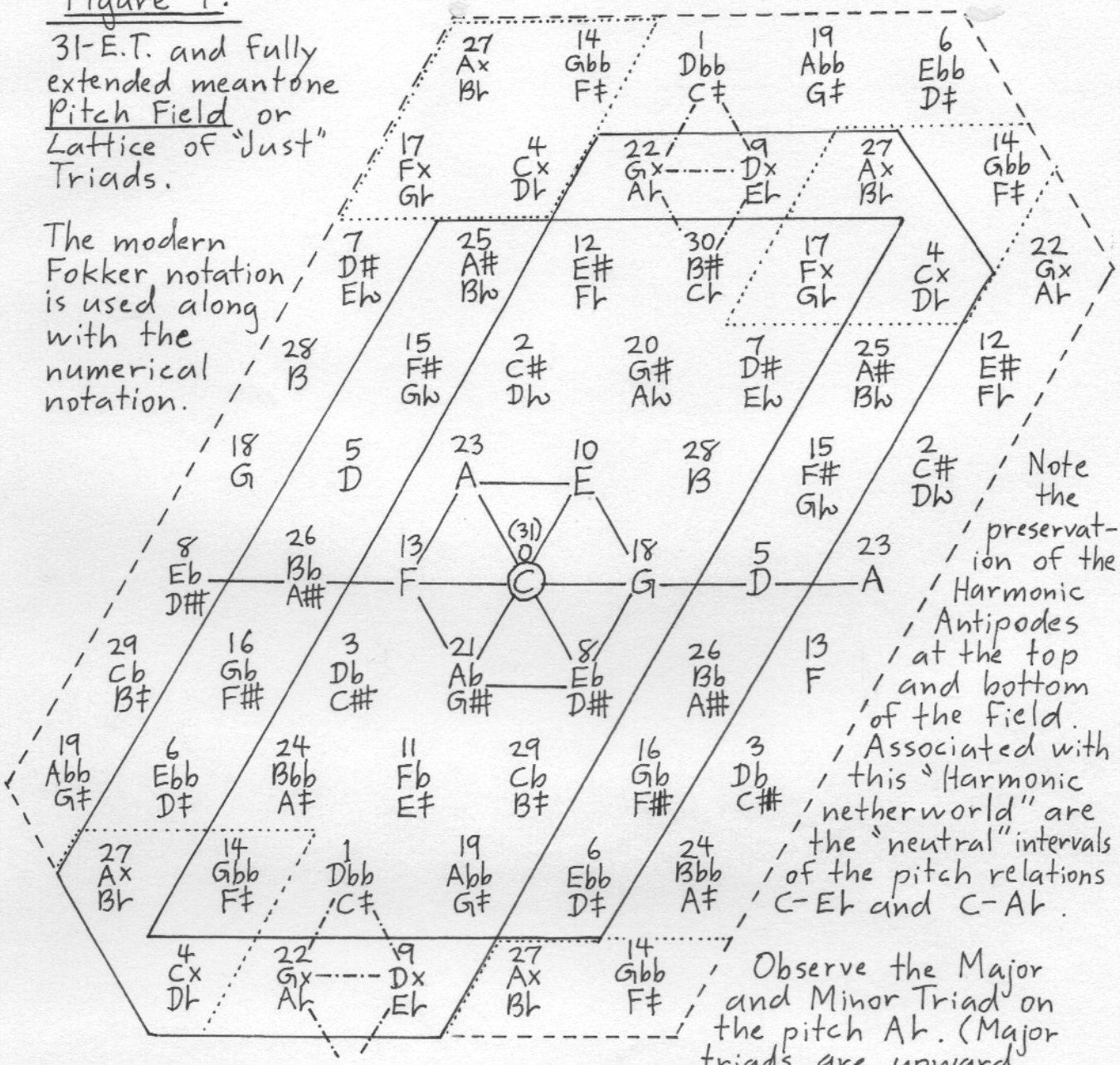


Tuning four pure Fifths from C results in /E which is a comma sharp of Just. Therefore flatten each Fifth by 1/4th Comma to maintain the natural musical Third.

Figure 4:

31-E.T. and fully extended meantone Pitch Field or Lattice of "Just" Triads.

The modern Fokker notation is used along with the numerical notation.



Note the preservation of the Harmonic Antipodes at the top and bottom of the Field. Associated with this "Harmonic netherworld" are the "neutral" intervals of the pitch relations C-E♭ and C-A♭.

Observe the Major and Minor Triad on the pitch A♭. (Major triads are upward pointing triangles, minor triads downward pointing). A♭ Major triad has pitches A♭-C♯-E♭, A♭ minor Triad A♭-C♭-E♭. The fifth A♭-E♭ has a unique relation to the generator C, witness the Diesis shifts C-C♯ and C-C♭. The fifth A♭-E♭ also "joins" or lies midway between the regions of the Antipodes. These two pitches also lie directly "opposite" the generator tone C on the circle of Fifths, and may be considered to inhabit the very centre of the Enharmonic Region.

Also note that two important boundaries of Just Intonation have dissappeared: the eastern-western boundaries associated with the comma and schisma intervals, and also the northern-southern boundaries associated with 53-E.T. and 19-E.T.

Figure 5: 31-E.T. Functional Field (plus numerical notation) with the 8 Regions indicated

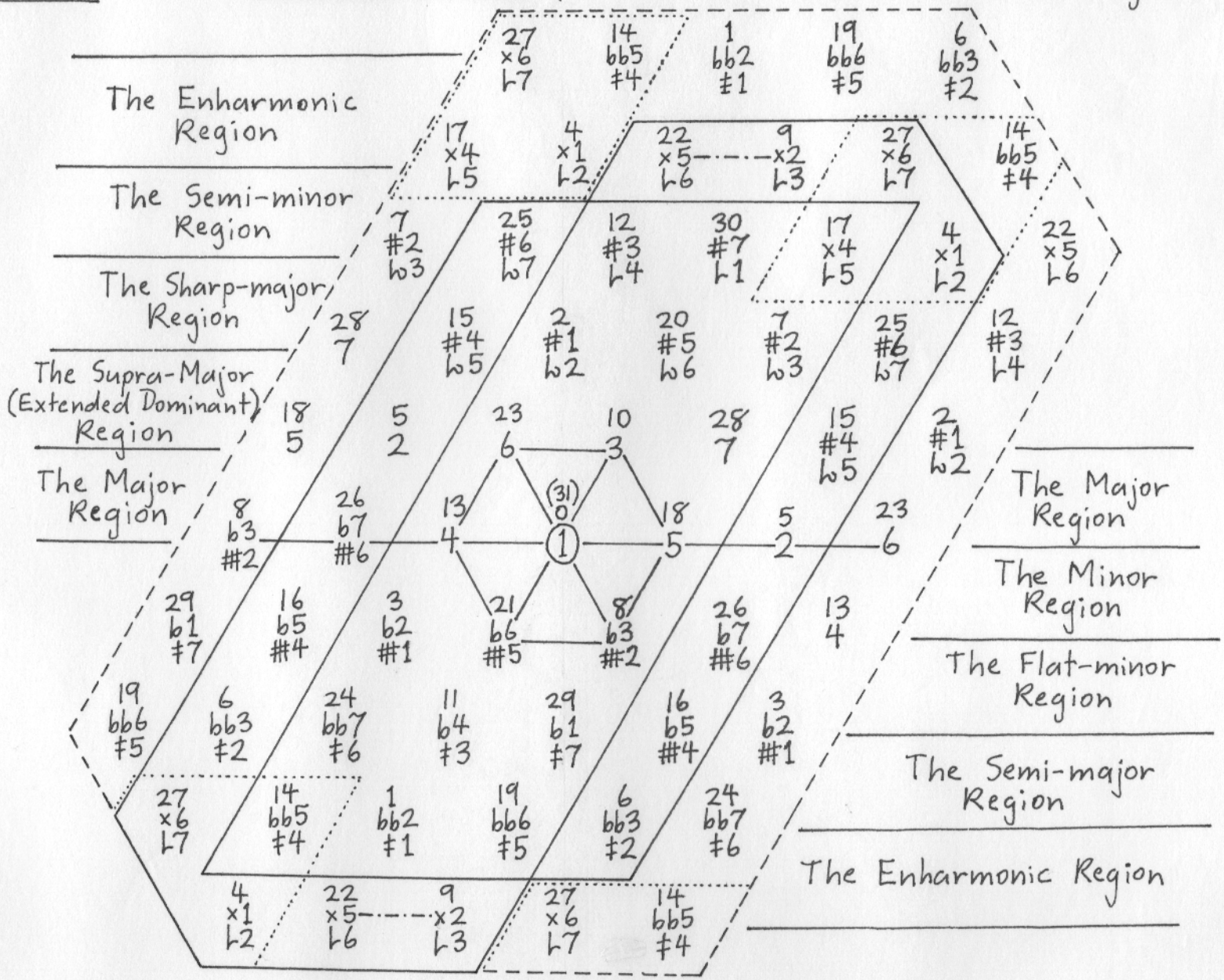


Figure 6:

The Eight Regions shown on the Circle of Fifths-Fourths. The Functional notation indicates Major Triads (outer part of the circle), and Minor Triads (inner part of the circle). The Majors and Minors are paired by the traditional relation known as "Relative Major-Relative Minor" (Type ③ pairing).

The Eight Regions are highlighted. Major-Minor Triad pairs associated with the boundary between regions (that is, Triads with two tones within the Meantone Corridor) are marked by asterisks:

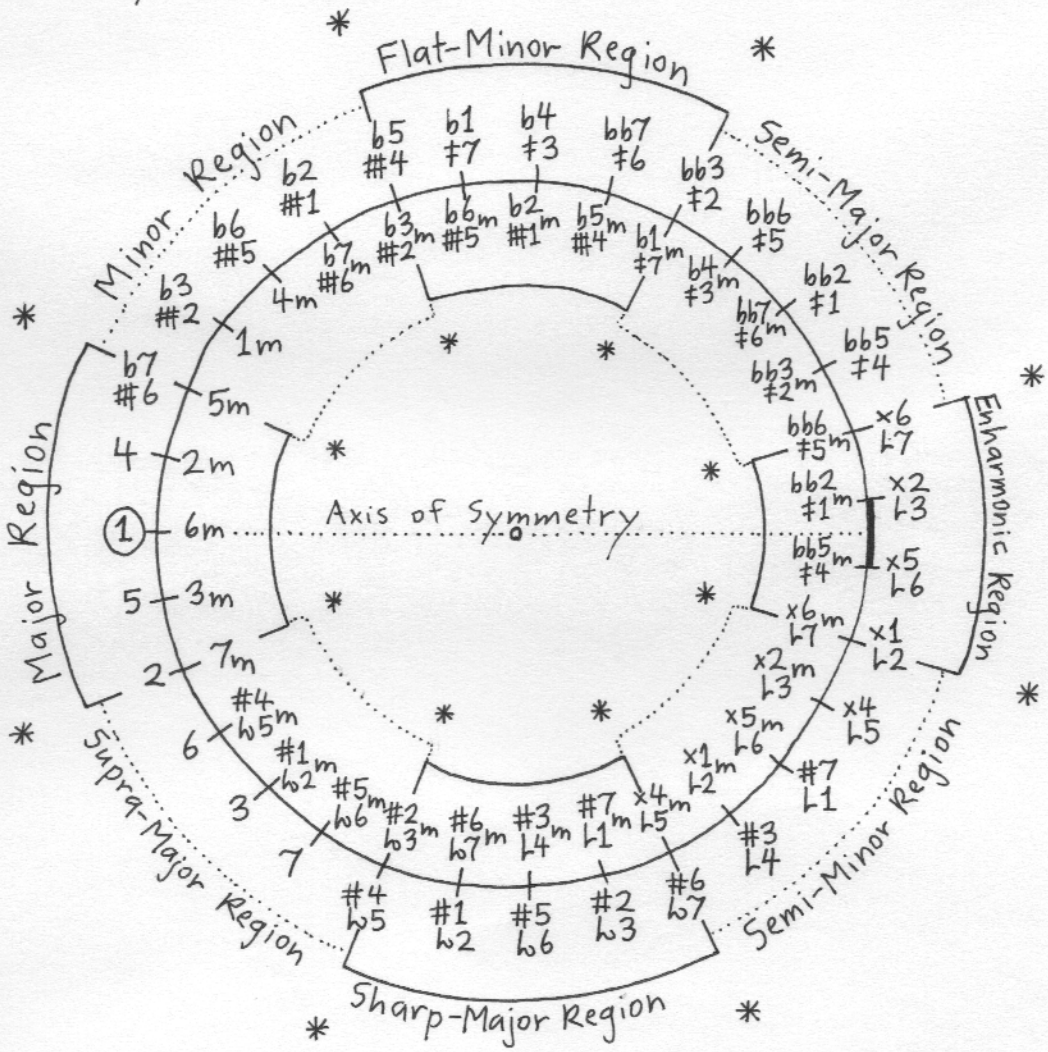


Figure 7:
Overview of Interval Classification:

Class	Diatonic Genus	Chromatic (Septimal) Genus	Enharmonic Genus	
①	0-Unison 31-Octave [1]			Perfect Consonances
②	18- Fifth [5] 13- Fourth [4]		19-"Wolf" Fifth [bb6] Doubleflat Sixth [♯5] Semisharp Fifth 12-"Wolf" Fourth [♯3] Sharp Third [ɻ4] Semiflat Fourth	Perfect Consonances
③	10- Major Third [3]	11- Septimal Major Third [♯3] Semisharp Third [b4] Flat Fourth	9- Neutral Third [ɻ3] Semiflat Third [x2] Doublesharp Second	Medial Consonances
③	23- Major Sixth [6]	24- Septimal Major Sixth [♯6] Semisharp Sixth [bb7] Double Flat Seventh		Medial Consonances
④	8- Minor Third [b3] Sesquisharp Second [♯2]	7- Septimal Minor Third [♯2] Sharp Second [ɻ3] Sesquiflat Third	22- Neutral Sixth [ɻ6] Semiflat Sixth [x5] Doublesharp Fifth	Medial Consonances
④	21- Minor Sixth [b6] Sesquisharp Fifth [♯5]	20- Septimal Minor Sixth [♯5] Sharp Fifth [ɻ6] Sesquiflat Sixth		Medial Consonances
⑤	5- Wholetone [2]	6- Septimal Wholetone [♯2] Semisharp Second [bb3] Doubleflat Third	4- Semiflat Wholetone [ɻ2] Doublesharp Tonic [x1]	Dissonances
⑤	26- Minor Seventh [b7] Flat Seventh [♯6] Sesquisharp Sixth	25- Septimal Minor Seventh [♯6] Sharp Sixth [ɻ7] Sesquiflat Seventh	27- Neutral Seventh [ɻ7] Semiflat Seventh [x6] Doublesharp Sixth	Dissonances
⑥	3- Diatonic Semitone [b2] Flat Second [♯1] Sesquisharp Tonic 28- Major Seventh [7]	2- Chromatic Semitone [♯1] Sharp Tonic [ɻ2] Sesquiflat Second 29- Flat Octave [b1] Semisharp Seventh [♯7]	1- Diesis, Doubleflat Second [♯1] Enharmonic [bb2] Semisharp Tonic 30- Semiflat Octave [ɻ1] Sharp Seventh [♯7]	Dissonances
⑦	16- Flat Tritone [b5] Flat Fifth [♯4] Sesquisharp Fourth	15- Sharp Tritone [♯4] Sharp Fourth [ɻ5] Sesquiflat Fifth	14- Semisharp Fourth [♯4] Doubleflat Fifth [bb5] 17- Semiflat Fifth [ɻ5] Doublesharp Fourth [x4]	Dissonances

Figure 8: Technical data on 31-E.T. intervals:

Numerical Notation	Interval Notation	Fractional Ratio	Frequency (1="C")	Ratio in Cents	Relation to Just Ratio
0=31	1	1.00	261.63 Hz.	0.0	1/1 – 0.0c.
1	♯1 · bb2	1.02261144	267.45	38.71	128/125 – 41.06c.
2	#1 · b2	1.04573415	273.595	77.42	25/24 – 70.67c.
3	##1 · b2	1.0693797	279.78	116.13	16/15 – 111.73c.
4	×1 · b2	1.09355991	286.11	154.84	
5	2	1.11828687	292.58	193.55	9/8 – 203.91c. 10/9 – 182.40
6	♯2 · bb3	1.14357294	299.19	232.26	8/7 – 231.17c.
7	#2 · b3	1.16943097	305.96	270.97	7/6 – 266.87c.
8	##2 · b3	1.19587327	312.88	309.68	6/5 – 315.64c.
9	×2 · b3	1.22291369	319.95	348.39	
10	3	1.25056552	327.185	387.10	5/4 – 386.31c.
11	♯3 · b4	1.2788426	334.58	425.81	32/25 – 427.34c.
12	#3 · b4	1.30775907	342.15	464.52	
13	4	1.33732938	349.885	503.23	4/3 – 498.045c.
14	♯4 · bb5	1.36756832	357.80	541.94	
15	#4 · b5	1.3984910	365.89	580.65	7/5 – 582.51c.
16	##4 · b5	1.43011289	374.16	619.35	10/7 – 617.49c.
17	×4 · b5	1.46244979	382.62	658.06	
18	5	1.49551788	391.27	696.77	3/2 – 701.955c.
19	♯5 · bb6	1.52933369	400.12	735.48	
20	#5 · b6	1.56391412	409.17	774.19	25/16 – 772.63c.
21	##5 · b6	1.59927646	418.42	812.90	8/5 – 813.69c.
22	×5 · b6	1.6354384	427.88	851.61	
23	6	1.67241801	437.55	890.32	5/3 – 884.36c.
24	♯6 · bb7	1.71023378	447.45	929.03	
25	#6 · b7	1.74890462	457.565	967.74	7/4 – 968.83c.
26	##6 · b7	1.78844987	467.91	1006.45	16/9 – 996.09c. 9/5 – 1017.6c.
27	×6 · b7	1.82888928	478.49	1045.16	
28	7	1.8702431	489.31	1083.87	15/8 – 1088.27c.
29	♯7 · b1	1.91253198	500.375	1122.58	48/25 – 1129.33c.
30	#7 · b1	1.95574707	511.69	1161.29	
31=0	1	2.00	523.25	1200.0	2/1 – 1200.0c.

The intervals, (in numerical notation), grouped with their inversions to form an octave-equivalent coupling – the 16 Interval Classes (One Monad – 15 Duads)

* 0 – 31	8 – 23
1 – 30	9 – 22
2 – 29	10 – 21
3 – 28	11 – 20
4 – 27	12 – 19
5 – 26	13 – 18
6 – 25	14 – 17
7 – 24	15 – 16

Each of the couples (ie. each Class) has a discrete Table of Functions, which defines every possible pattern exhibiting <u>that</u> particular internal relationship.

Figure 9: The difference in interval size between 31-E.T. and the extended Quarter-comma Meantone Tuning: (All measurements are in Cents)

Interval Number Notation	Interval Functional Notation	31-E.T.	"Pure" Meantone	Difference M-31	
22-	×5·b6	851.61 c.	854.74 c.	+ 3.13	
9-	×2·b3	348.39 c.	351.315 c.	2.925	
27-	×6·b7	1045.16 c.	1047.89 c.	2.73	
14-	♯4·bb5	541.94 c.	544.47 c.	2.53	
1-	♯1·bb2	38.71 c.	41.05 c.	2.34	
19-	♯5·bb6	735.48 c.	737.63 c.	2.15	
6-	♯2·bb3	232.26 c.	234.21 c.	1.95	
24-	♯6·bb7	929.03 c.	930.79 c.	1.76	
11-	♯3·b4	425.81 c.	427.37 c.	1.56	Sub-dominant direction
29-	♯7·b1	1122.58 c.	1123.95 c.	1.37	
16-	#4·b5	619.35 c.	620.53 c.	1.18	
3-	#1·b2	116.13 c.	117.105 c.	0.975	↑
21-	#5·b6	812.90 c.	813.68 c.	0.78	
8-	#2·b3	309.68 c.	310.26 c.	0.58	
26-	#6·b7	1006.45 c.	1006.84 c.	0.39	
13-	4	503.23 c.	503.42 c.	0.19	
0=31-	1	0.0 c.	0.0 c.	0.0 c.	Tonic
18-	5	696.77 c.	696.58 c.	0.19	
5-	2	193.55 c.	193.16 c.	0.39	
23-	6	890.32 c.	889.74 c.	0.58	
10-	3	387.10 c.	386.32 c.	0.78	
28-	7	1083.87 c.	1082.895 c.	0.975	↓
15-	#4·b5	580.65 c.	579.47 c.	1.18	Dominant Direction
2-	#1·b2	77.42 c.	76.05 c.	1.37	
20-	#5·b6	774.19 c.	772.63 c.	1.56	
7-	#2·b3	270.97 c.	269.21 c.	1.76	
25-	#6·b7	967.74 c.	965.79 c.	1.95	
12-	#3·b4	464.52 c.	462.37 c.	2.15	
30-	#7·b1	1161.29 c.	1158.95 c.	2.34	
17-	×4·b5	658.06 c.	655.53 c.	2.53	
4-	×1·b2	154.84 c.	152.11 c.	2.73	
22-	×5·b6	851.61 c.	848.685 c.	2.925	
9-	×2·b3	348.39 c.	345.26 c.	−3.13	

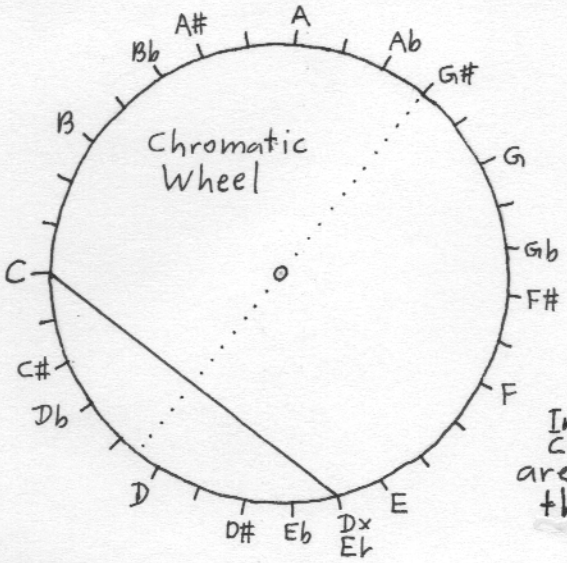
Meantone Fifth = ratio $\sqrt[4]{5} = 1.4954 = 696.579$ cents

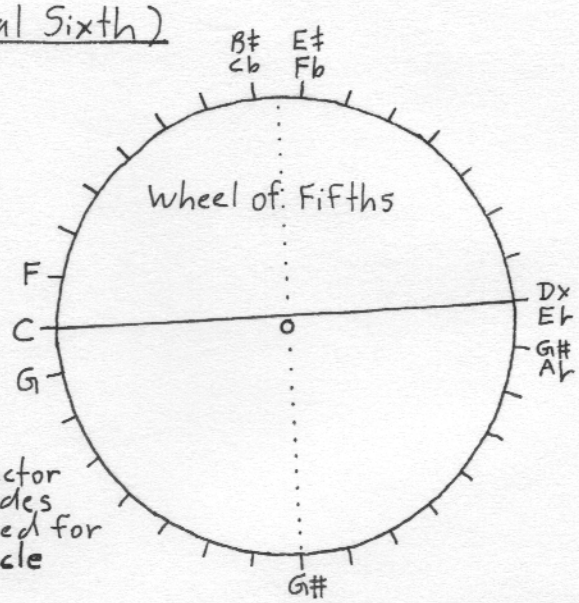
The "pure" meantone tuning can be expressed as a line of fifths in which the ratios are all multiples of two and five. Below is a sample of part of the line (in Functional notation) with appropriate ratios:

....b4	—b1	—b5	—b2	—b6	—b3	—b7	—4	—1	—5	—2	—6	—3	—7	—#4	—#1	—#5.....
$\frac{2^5}{5^2}$	$\frac{2^4\cdot\sqrt[4]{5}}{5^2}$	$\frac{2^4}{5\cdot\sqrt{5}}$	$\frac{2^3}{5\cdot\sqrt[4]{5}}$	$\frac{2^3}{5}$	$\frac{2^2\cdot\sqrt[4]{5}}{5^2}$	$\frac{2^2}{\sqrt{5}}$	$\frac{2}{\sqrt[4]{5}}$	1/1	$\frac{\sqrt[4]{5}}{1}$	$\frac{\sqrt{5}}{2}$	$\frac{5}{2\cdot\sqrt[4]{5}}$	$\frac{5}{2^2}$	$\frac{5\cdot\sqrt[4]{5}}{2^2}$	$\frac{5\cdot\sqrt{5}}{2^3}$	$\frac{5^2}{2^3\cdot\sqrt[4]{5}}$	$\frac{5^2}{2^4}$

Figure 10: A sample from the book of Functional Tables for 31-E.T.

Duad 9-22 (Neutral Third-Neutral Sixth)





Interval Vector Colour Codes are ommited for **this article**

0	9	– Primary Function
1	10	–
2	11	– Prominant Function
3	12	– Prominant Function
4	13	–
5	14	–
6	15	–
7	16	– Prominant Function
8	17	–
9	18	–
10	19	–
11	20	– Prominant Function
12	21	– Prominant Function
13	22	–
14	23	–
15	24	–
16	25	– Prominant Function
17	26	–
18	27	–
19	28	–
20	29	– Prominant Function
21	30	– Prominant Function
22	0	– Primary Function
23	1	–
24	2	–
25	3	– Prominant Function
26	4	–
27	5	–
28	6	–
29	7	– Prominant Function
30	8	– Prominant Function

The two Wheels show representative example C-Eb of this Class of dissonant Duads.

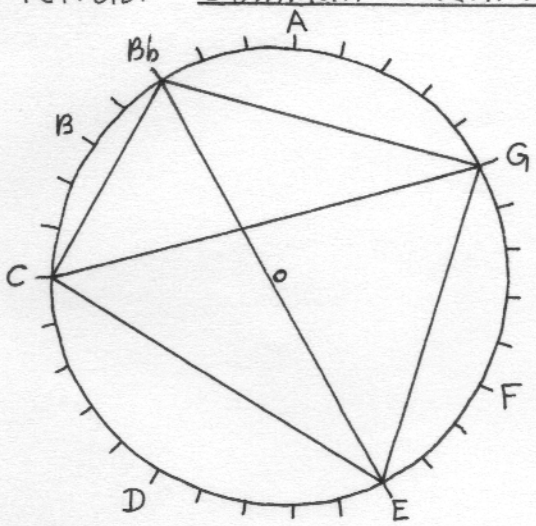
The pattern is symmetrical about the pitch G#. The symmetry also appears on the Wheel of Fifths. Note the remote position of the interval on the Wheel of Fifths - indicating dissonance.

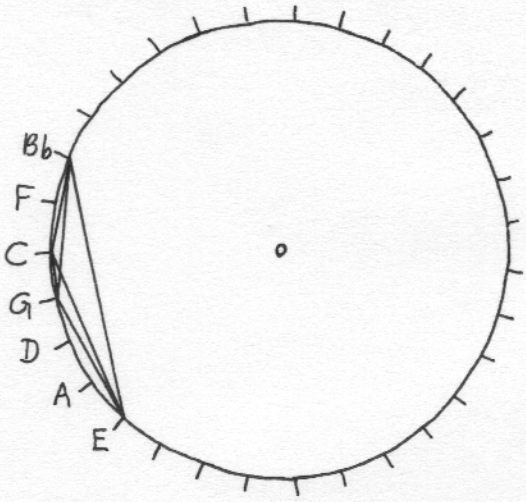
Certain functions on the table are marked as 'prominant.' Only these functions display a pairing of a sharp and a flat (ie. 11-20 means b4-#5). All other functions involve semi-flats or semi-sharps, for instance 18-27 (or 5-b7).

In fact, all of the Duad Classes (the interval harmonies) embody an Axis of symmetry which centres on a definite pitch in the system. More complex harmonies (eg. Triads, Tetrads, Pentads, etc.) are either symmetrical or non-symmetrical. Those which are not symmetrical have "flip-over" compliment, forming pairs (eg. Major Triad + Minor Triad).

Figure 11: A sample Tetrad from the book of Functional Tables for 31-E.T.

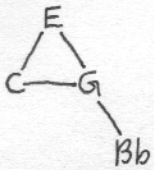
Tetrad - "Dominant Seventh" Type Tetrad (Diatonic Genus)

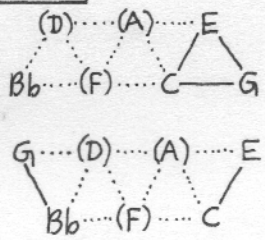




Functions:				Typical Resolution
0	10	18	26	(13)-Primary Function-Resolve to Sub-dominant
1	11	19	27	(14)-
2	12	20	28	(15)-
3	13	21	29	(16)-
4	14	22	30	(17)-
5	15	23	0	(18)-Primary Function-"Dominant Preparation" resolve to Dominant
6	16	24	1	(19)-
7	17	25	2	(20)-
8	18	26	3	(21)-
9	19	27	4	(22)-
10	20	28	5	(23)-"Secondary Dominant"
11	21	29	6	(24)-
12	22	30	7	(25)-
13	23	0	8	(26)-Primary Function
14	24	1	9	(27)-
15	25	2	10	(28)-
16	26	3	11	(29)-
17	27	4	12	(30)-
18	28	5	13	(0)-The Dominant-resolves to Tonic
19	29	6	14	(1)-
20	30	7	15	(2)-
21	0	8	16	(3)-Primary Function
22	1	9	17	(4)-
23	2	10	18	(5)-"Secondary Dominant"
24	3	11	19	(6)-
25	4	12	20	(7)-
26	5	13	21	(8)-
27	6	14	22	(9)-
28	7	15	23	(10)-"Secondary Dominant"
29	8	16	24	(11)-
30	9	17	25	(12)-

Pattern on the Field Diagram:

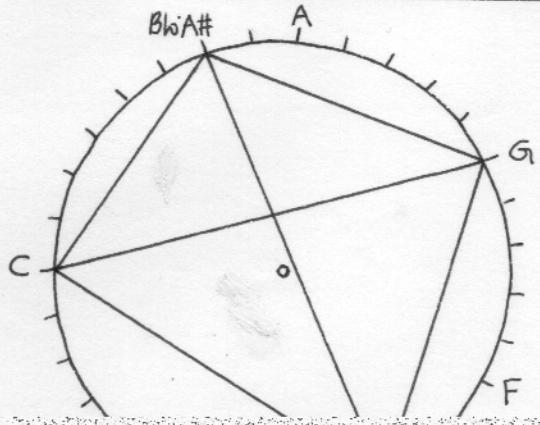




We can see (from examining the wheels) that this familiar harmony is not symmetrical. The Function in brackets to the right of the pattern function is the most usual resolution. For instance, $C^7\square$ resolves to F Major $\triangle$. Note, however, that another natural resolution is to Dm$\triangle$ or Am$\triangle$ (as seen from the pattern on the Field Diagram and from the Wheel of Fifths).

Figure 12: A sample Tetrad from the book of Functional Tables for 31-E.T.

Tetrad - "Harmonic Series Dominant Seventh" Tetrad - (Chromatic Genus)



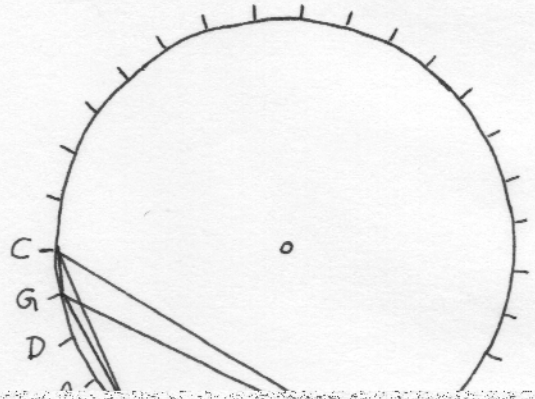
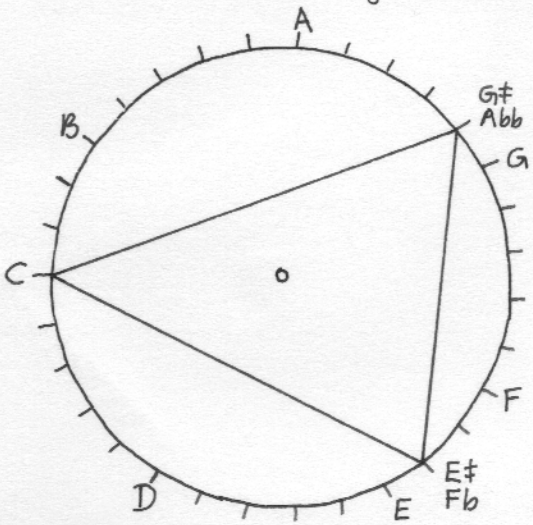
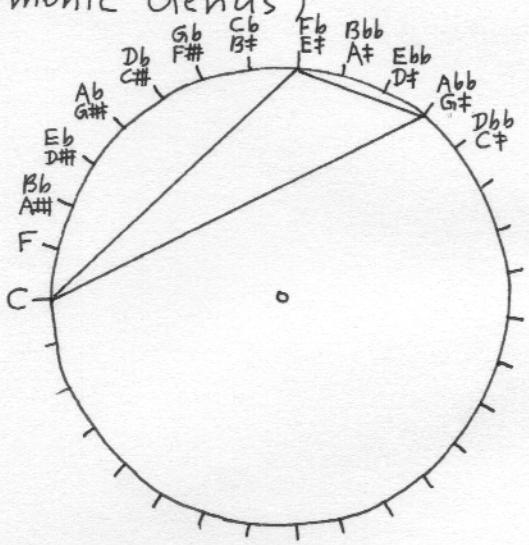


Figure 13: A sample Triad from the book of Functional Tables for 31-E.T.

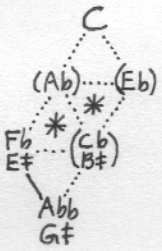
Triad - "Wolf" Major Triad (Enharmonic Genus)





M		m	
0	11	19	(21) - Primary Function
1	12	20	(22) -
2	13	21	(23) - Prominant Function
3	14	22	(24) -
4	15	23	(25) -
5	16	24	(26) -
6	17	25	(27) -
7	18	26	(28) - Prominant-Dominant Prep.
8	19	27	(29) -
9	20	28	(30) -
10	21	29	(0) - Prominant Function
11	22	30	(1) -
12	23	0	(2) - Primary Function
13	24	1	(3) -
14	25	2	(4) -
15	26	3	(5) - Prominant Function
16	27	4	(6) -
17	28	5	(7) -
18	29	6	(8) -
19	30	7	(9) -
20	0	8	(10) - Primary Function
21	1	9	(11) -
22	2	10	(12) -
23	3	11	(13) - Prominant Function
24	4	12	(14) -
25	5	13	(15) - Prominant Function
26	6	14	(16) -
27	7	15	(17) -
28	8	16	(18) - Prominant Function
29	9	17	(19) -
30	10	18	(20) -

Pattern on the Field Diagram

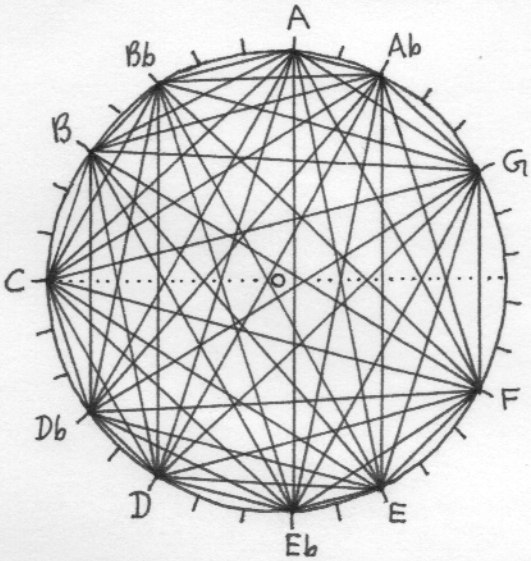


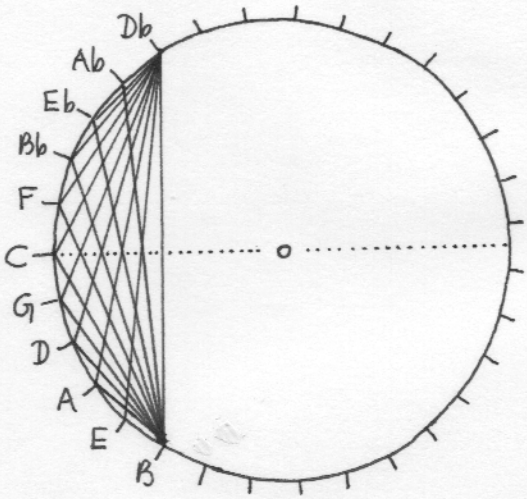
The natural resolution for this dissonance is marked by the asterisks; that is, the C Wolf Major Triad naturally moves to either Ab minor Triad or to Fb Major Triad. These resolutions are also marked (M and m) on the Functional Table. It must be emphasized, however, that this resolution pattern is not the only one. Other paths suggest themselves from the Wheel of Fifths (within the region bounded by C and G♯). Thus even dissonant structures such as this have characteristic resolutions inferred from the Field Diagram and Wheel of Fifths.

Note that the change of one tone (C to C♯) transforms the Wolf Major Triad into a Just Major Triad (C♯-E♯-G♯). Here is a very "non-traditional" form of resolution.

Figure 14: A sample pattern from the book of Functional Tables for 31-E.T.

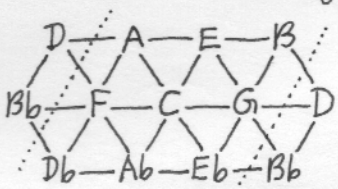
A Chromatic Undecad (11 note pattern):





*											
0	3	5	8	10	13	18	21	23	26	28	–Prime
1	4	6	9	11	14	19	22	24	27	29	–
2	5	7	10	12	15	20	23	25	28	30	–
3	6	8	11	13	16	21	24	26	29	0	–Prime
4	7	9	12	14	17	22	25	27	30	1	–
5	8	10	13	15	18	23	26	28	0	2	–Prime
6	9	11	14	16	19	24	27	29	1	3	–
7	10	12	15	17	20	25	28	30	2	4	–
8	11	13	16	18	21	26	29	0	3	5	–Prime
9	12	14	17	19	22	27	30	1	4	6	–
10	13	15	18	20	23	28	0	2	5	7	–Prime
11	14	16	19	21	24	29	1	3	6	8	–
12	15	17	20	22	25	30	2	4	7	9	–
13	16	18	21	23	26	0	3	5	8	10	–Prime
14	17	19	22	24	27	1	4	6	9	11	–
15	18	20	23	25	28	2	5	7	10	12	–
16	19	21	24	26	29	3	6	8	11	13	–
17	20	22	25	27	30	4	7	9	12	14	–
18	21	23	26	28	0	5	8	10	13	15	–Prime
19	22	24	27	29	1	6	9	11	14	16	–
20	23	25	28	30	2	7	10	12	15	17	–
21	24	26	29	0	3	8	11	13	16	18	–Prime
22	25	27	30	1	4	9	12	14	17	19	–
23	26	28	0	2	5	10	13	15	18	20	–Prime
24	27	29	1	3	6	11	14	16	19	21	–
25	28	30	2	4	7	12	15	17	20	22	–
26	29	0	3	5	8	13	16	18	21	23	–Prime
27	30	1	4	6	9	14	17	19	22	24	–
28	0	2	5	7	10	15	18	20	23	25	–Prime
29	1	3	6	8	11	16	19	21	24	26	–
30	2	4	7	9	12	17	20	22	25	27	–

Pattern on Field Diagram:

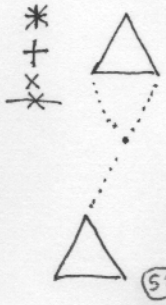


This pattern is an 11 note segment of the circle of Fifths, stopping short of the 12th tone which introduces the dissonant (Wolf) Fifth. The pattern is symmetrical. Note that there are 11 Primary Functions, which become the 11 "modes" of the pattern. In 31-E.T. complex structures, like chromatic scales or even more complex patterns, have "modes." The many subsets of this harmony include Harmonic Series Seventh Tetrad (Db-F-Ab-B), as well as two sub-minor triads (on Db and Ab, eg. Ab-B-Eb) an also two Super-Major (Chromatic Genus) Triads, on A and E).

54

I − bb$\underline{\text{III}}$ · ♯$\underline{\text{II}}$m

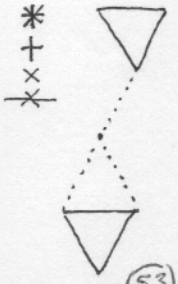
Im − #$\underline{\text{VI}}$ · w$\underline{\text{VII}}$



52

I − #$\underline{\text{III}}$ · ⊢$\underline{\text{IV}}$

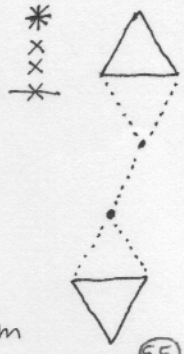
I − bb$\underline{\text{VI}}$ · ♯$\underline{\text{V}}$



53

Im − #$\underline{\text{III}}$ · ⊢$\underline{\text{IV}}$m

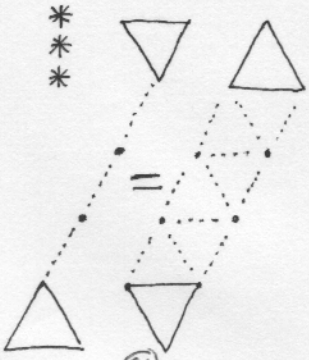
Im − bb$\underline{\text{VI}}$ · ♯$\underline{\text{V}}$m



55

I − bb$\underline{\text{VI}}$ · ♯$\underline{\text{V}}$m

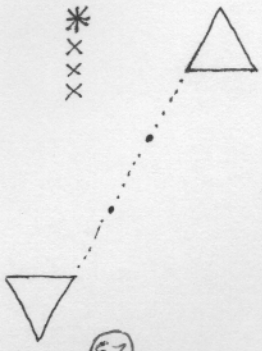
Im − #$\underline{\text{III}}$ · ⊢$\underline{\text{IV}}$



56

I − bb$\underline{\text{II}}$ · ♯I m

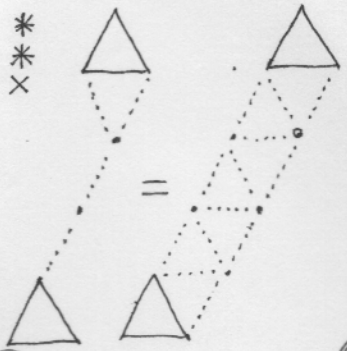
Im − #$\underline{\text{VII}}$ · ⊢I



57

I − bb$\underline{\text{V}}$ · ♯$\underline{\text{IV}}$m

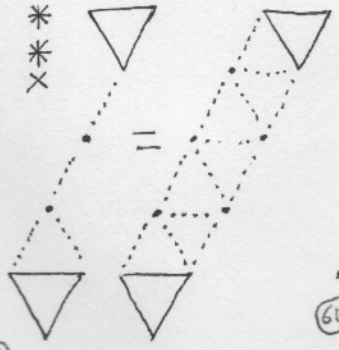
Im − ×$\underline{\text{IV}}$ · ⊢$\underline{\text{V}}$



58

I − ×$\underline{\text{II}}$ · ⊢$\underline{\text{III}}$

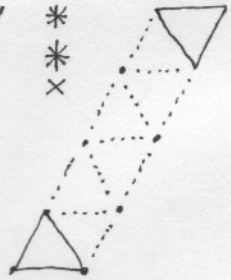
I − ×$\underline{\text{V}}$ · ⊢$\underline{\text{VI}}$



59

Im − ×$\underline{\text{II}}$ · ⊢$\underline{\text{III}}$m

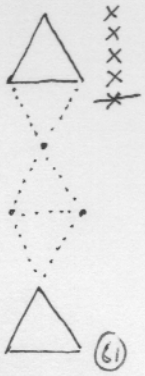
Im − ×$\underline{\text{V}}$ · ⊢$\underline{\text{VI}}$m



60

I − ×$\underline{\text{II}}$ · ⊢$\underline{\text{III}}$m

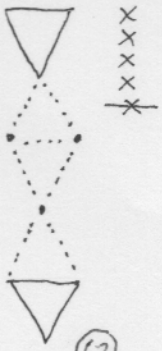
Im − ×$\underline{\text{V}}$ · ⊢$\underline{\text{VI}}$



61

I − ×I · ⊢$\underline{\text{II}}$

I − ×$\underline{\text{VI}}$ · ⊢$\underline{\text{VII}}$



62

Im − ×I · ⊢$\underline{\text{II}}$m

Im − ×$\underline{\text{VI}}$ · ⊢$\underline{\text{VII}}$m

63

I − ×$\underline{\text{VI}}$ · ⊢$\underline{\text{VII}}$m

Im − ×I · ⊢$\underline{\text{II}}$