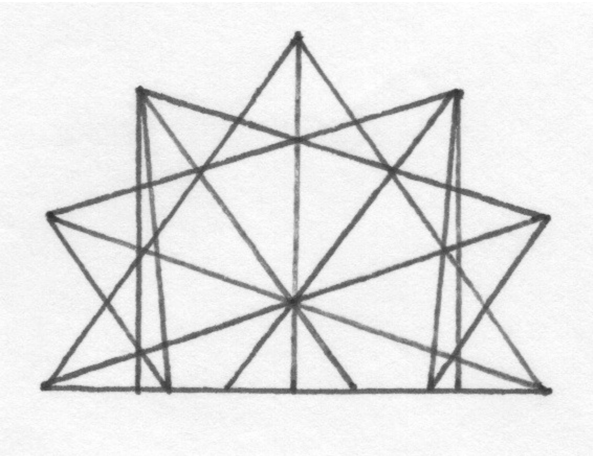
An Arithmetical Rubric

attending the distribution of 'best' multiple-divisions



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An **ARITHMETICAL RUBRIC** attending the distribution of 'best' multiple-divisions:

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This sequence of ratios suggests the most significant equal-tempered resolution-levels of the field of harmony. The information is presented in a table which can be extended indefinitely. Here is how to read the table. The left-hand column gives the *Source Ratio*, an archetypal series of epimoric (superparticular) ratios: 1:2, 2:3, 3:4, and so on. These ratios then generate another series of epimoric ratios, entitled the *Significant Relation*. The method of generation will be explained below. This second column consists of a number of highly important musical ratios which approximate the step-size of a superior musical temperament system. The column on the right gives the associated *Prominant et* (et stands for equal temperament). The other columns factor the *Significant Relations* in order to clarify various interesting numerical properties inherent in this progression.

Source Ratio	Significant Relation	Factoring to show Triangular Numbers	Further Decomposition	Prominant et
1:2	8:9	$8 * 1 : 3^2$	$8 * 1 * 1 : 3^2$	6? 7?
2:3	24:25	$8 * 3 : 5^2$	$8 * 1 * 3 : 5^2$	19
3:4	48:49	$8 * 6 : 7^2$	$8 * 2 * 3 : 7^2$	31
4:5	80:81	$8 * 10 : 9^2$	$8 * 2 * 5 : 9^2$	53
5:6	120:121	$8 * 15 : 11^2$	$8 * 3 * 5 : 11^2$	77
6:7	168:169	$8 * 21 : 13^2$	$8 * 3 * 7 : 13^2$	118
7:8	224:225	$8 * 28 : 15^2$	$8 * 4 * 7 : 15^2$	152
8:9	288:289	$8 * 36 : 17^2$	$8 * 4 * 9 : 17^2$	205
9:10	360:361	$8 * 45 : 19^2$	$8 * 5 * 9 : 19^2$	239
10:11	440:441	$8 * 55 : 21^2$	$8 * 5 * 11 : 21^2$	289
11:12	528:529	$8 * 66 : 23^2$	$8 * 6 * 11 : 23^2$	357
12:13	624:625	$8 * 78 : 25^2$	$8 * 6 * 13 : 25^2$	441
13:14	728:729	$8 * 91 : 27^2$	$8 * 7 * 13 : 27^2$	506
14:15	840:841	$8 * 105 : 29^2$	$8 * 7 * 15 : 29^2$	612
15:16	960:961	$8 * 120 : 31^2$	$8 * 8 * 15 : 31^2$	665

The *Significant Relation* ratios were derived from the *Source Ratios* by taking the HARMONIC MEAN (HM) and the ARITHMETIC MEAN (AM) and relating them to each other. For example, let us use 4:5 as our *Source Ratio*. The HM is ratio 8:9, and the AM is ratio 9:10. These two ratios relate to each other by the ratio 80:81 (the syntonic comma). Although Archytas surrounded this procedure with a haze of mystical complexity, the derivation is absurdly simple. In order to find the HM and the AM of a given ratio, double the numbers and add the middle term. Thus 4:5 becomes 8:10 and then 8:9:10. The first part (8:9) is the HM, the second part (9:10) is the AM. This procedure is fundamental to monochord arithmetic, as demonstrated in the writings of Ptolemy and other ancient writers. The third classical *mean*, the GEOMETRIC MEAN, is not used here, since it generates an irrational ratio not relevant to our procedure.

One can legitimately ask the question: why choose 53-et as the relevant temperament rather than, say, 50-et or 55-et, which also approximate a comma in their step-size? Although this is true, 53-et is the best of the three choices, since it has special properties lacking in the other two. The other two are 'secondary' choices for a comma-resolution temperament. The same situation holds in other cases as well. For example, we could consider 34-et as an 'auxiliary' of 31-et in the diesis-resolution, or 72-et instead of 77-et. In order to make an intelligent choice, one must examine the deviation levels of all these 'neighbourhood' systems, and choose the best one. This situation alerts us to the fact that our procedure is not rigorously isomorphic. The *Significant Relation* does not unambiguously define the tempered system; rather, it warns us of the likelihood that there is a good system 'in the neighbourhood.' A mathematical rubric which unambiguously defines the distribution of 'best' systems does not exist, although many attempts have been made to find it by using continued fractions. Such a rubric is probably related to the distribution of prime numbers, one of the great unsolved problems of mathematics.

I found this ratio progression when working with traditional monochord procedures. The orderly emanation of complexity out of the original unity is a mainstay of classical metaphysics, which has a strong musical 'bias.' When I factored the *Significant Relation* ratios further structural features became apparent, which also connect with archaic musical mathematics. For example, various ancient mathematicians noted a peculiar property of odd numbers: every odd number above 1, when squared, results in a multiple of 8 plus 1. This property has obvious relevance for our table. Also, when the number 8 is 'removed' from the left-hand side of the ratio, the remaining portion forms the TRIANGULAR NUMBER SEQUENCE (1, 3, 6, 10, 15 etc.) which underlies the *Tetractys pattern* of the Pythagoreans. This discovery gave me much aesthetic satisfaction, but it is not surprising. Triangular numbers keep recurring in various musical contexts, not least in the quantification of relatedness between harmonic events. Two events have one relation (or interval) between them, three events have three relations, four events six relations, and so on. The prominance of the number 8 is also peculiarly musical. For the number 8 defines the 'boundary' between consonance and dissonance. The ratio 7:8 sits on the 'border' between the two 'realms,' 6:7 being definitely consonant, while 8:9 is definitely dissonant. Ratios with constituent numbers higher than 8 are classed and experienced as dissonant. On a higher architectonic level, 3-Limit and 5-Limit harmony form the traditional 'store-house' of strong tuning *norms*, with 7-Limit harmony acting as an extension of those norms. The 'leap' into the 11-Limit or higher primes is an incursion into somewhat 'alien territory.' Again, 'eightness' forms a natural boundary within harmonic 'space.'

Notice that our modern system of temperament, 12-et, is missing from the above table. While musing over this state of affairs, I found a secondary progression based on the squares of the even numbers instead of the odd numbers. This 'companion' to the other table does not have quite the 'status' of the first table, since the *Source Ratios* are no longer epimoric. (I am adopting Ptolemy's aesthetic stance in which epimoric ratios have an elevated status over non-epimores). However, the *Significant Relation* ratios are still epimoric, and this progression also suggests some very important systems of temperament.

It also embodies some special numerical properties which I leave the reader to ponder. We may call this table a '2nd order' progression of significant ratios:

Source Ratio	Significant Relation	Factors	Prominant et
1:3	3:4	$1 * 3 : 2^2$	2?
3:5	15:16	$3 * 5 : 4^2$	12
5:7	35:36	$5 * 7 : 6^2$	22
7:9	63:64	$7 * 9 : 8^2$	43
9:11	99:100	$9 * 11 : 10^2$	65
11:13	143:144	$11 * 13 : 12^2$	99
13:15	195:196	$13 * 15 : 14^2$	137
15:17	255:256	$15 * 17 : 16^2$	171
17:19	323:324	$17 * 19 : 18^2$	224
19:21	399:400	$19 * 21 : 20^2$	270
21:23	483:484	$21 * 23 : 22^2$	323

Again, this table has ambiguities concerning the co-relations. For example, 43-et could be replaced by 41-et, also 99-et could be replaced by 94-et. Yet between these two tables, practically all of the best systems of equal-temperament are covered.

We could go on to generate a '3rd order' progression whose first *Source Ratio* would be 1:4. However, the status of such a table is greatly diminished, since the *Significant Relation* ratios are no longer epimoric. An examination of the factors in this progression reveals that complexity is fast overtaking the beautiful numerical symmetries of the above two tables. In addition, the equal-temperaments suggested by these ratios are no longer of the first rank. It appears that we have come to an end in this method of investigation.

These two tables offer an intriguing 'family' of *Significant Relation* ratios which have great importance within the matrix of just-intonation harmony. Many of these ratios have been given specific names in honor of their place within harmonic structure. For example, 8:9 is the major whole-tone so well approximated by 12-et, and a mainstay of 3-Limit harmony. 24:25 is the small 5-Limit chromatic semitone. 48:49 is the 7-Limit enharmonic 'quarter-tone' mentioned in Plato's *Republic*. 80:81, of course, is the famous syntonic comma which is ubiquitous whenever an extended 5-Limit matrix is generated. 120:121 is an important 11-Limit ratio which is often seen when an 11-Limit matrix is integrated into the traditional 5-Limit. 224:225 is called the 7-Limit kleisma, and appears whenever the 7-Limit matrix is integrated into the 5-Limit. In the second table, 3:4 needs no introduction. 15:16 is the 5-Limit diatonic semitone. 35:36 is the 7-Limit diesis beloved of Archytas. 63:64 is the 7-Limit comma. And so on.

Perhaps the structural prominance of these epimoric ratios then suggests tempered resolutions which have special properties. At any rate, this interesting co-relation between epimoric ratios and temperaments points to some deep structural feature of harmony whose causes and wider implications as yet elude me.