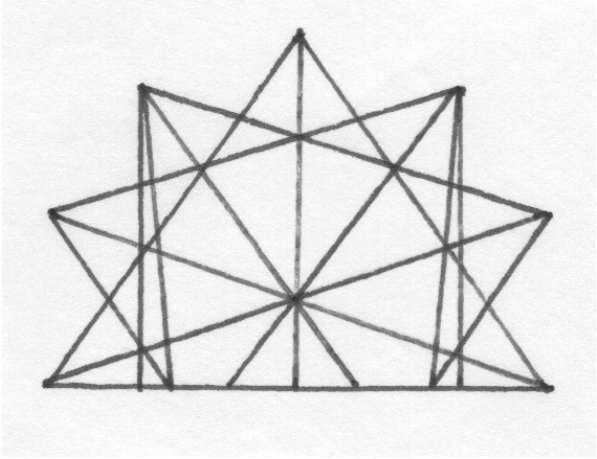
A short-list of Musical-Cosmological Monochords



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'holds' 3 in her lap.

The use of the process of 'doubling' in arithmetic is very ancient. Within the oldest strata of Egyptian arithmetic, there lies a method for making use of the principle that every integer can be expressed as the sum of numbers 2^n. They use this principle to calculate the higher multiple of any number as the sum of numbers arising from consecutive doublings. By using this technique, they did not need a multiplication table. Of course, the multiplication table (which is itself a potent source for ratio generation) also existed from ancient times. In Babylonia, they used the multiplication table for 60*60 as well as 10*10. The point here is that the significance of the process of doubling cannot be underestimated in ancient mathematics, especially in the context of the monochord.

All of these monochords must be seen as the expression of the same octave matrix in different numerical disguises. The varying 'numerosity' of the sequences stems purely from the desire for a least common multiple, which we require in order to integrate some level of field complexity. The underlying principle is that of economy--we always use the smallest integers possible in order to express the harmony as a natural number sequence within the desired context. The empirical structure of musical harmony itself imposes an hierarchical order into the field of numbers. The monochords given here are the 'cream of the crop;' that is, they are the most significant ones from the stand-point of field structure. Rigorously following only musical significance as our guide, we also then generate number sequences which dominate ancient cosmology.

Although this material can be approached as a purely arithmetical 'game,' the ancients saw great significance in this exploration. By all accounts, they exhibited cultures in which music played an important role. The Babylonians were blessed with "the greatest system of musical ritual in any ancient religion." The ancient Egyptians had many varieties of harps. Moreover, musical tuning was closely aligned with the other sciences, especially calendrics. Endless allegories were generated using the evidence of the monochord as 'justification.'

For example, in the Ṛg Veda, the One (the original Unity) is hermaphroditic, but it produces a virgin daughter (2) without the benefit of another parent. The number 2 is female (the first manifestation of gender) because it is the mother or matrix of all subsequent 'children.' The numbers 1 and 2 now 'couple' in divine incest to generate the prime number 3. Here we have the first male number, the 'divine male number.' The number 3 then becomes the 'demiurge' or subordinate god who generates the 'universe,' for it generates endless new tones until we give it some limit of expansion. The virgin daughter (2) now 'copulates' with the divine male number (3) to give the divine 'human' prime number 5. We now have the ancient sacred trinity of gods who generate all of the numbers in this study, since every number herein can be 'decomposed' into a unique product of the

primes 2, 3, and 5. These numbers are the ultimate 'parents' of all these sequences.

Even before we actually divide the monochord into a sequence, it contains the whole complexity of harmonic relations in potential. This pre-manifestation is called ASAT in the Ṛg Veda, and can mean both silence and the noise and confusion of the 'plenum.' It is the state before order is 'distilled' from chaos. In all of the ancient creation myths, order is created as an 'island' in the sea of chaos, the Primeval Waters. The initial act is seen as the separation of the Earth (2) from the Sky (1), since the cyclical nature of the octave is the invariant ground of all tuning systems. All harmonies are 'children' of the initial division.

The sexual metaphor is consistent throughout ancient mythology concerning numbers. Odd numbers, and especially prime numbers, are male. They unite through 'marriages' with even numbers to generate tone children. Simple rational numbers have the priority as strong harmonies which are parents to greater numerosity. The powers of the numbers 3 and 5 are the 'heroes' who 'conquer' the pitch field. In the Vedas, Indra conquers the dragon Vṛtra, dismembering it to create the universe. The dragon is a metaphor for the Asat, the potential complexity from which harmony is distilled. Even though it is conquered, the 'conflict' never ends, since the hero himself is totally dependent upon, indeed, is a crystallized aspect of the continuum itself. Without the Asat, there can be no Indra or any of the other gods.

The metaphor of the 'dragon' is found in all of the ancient cultures. The Vedic name Vṛtra means 'to spin,' to proliferate numerosity. In Babylonia, Marduk dismembers the female serpent Tiamat, from whose body he fashions the cosmos. In Egypt we have the eternal conflict between Osiris and Set. Out of the universe of sheer vibration, energy, and possibility, the world with its gods and men emerges. Harmony is a precious distillate of vibratory-experience-in-general. We can expand this simple matrix of relations to encompass more and more interval complexity, until the extent of the matrix itself approaches the complexity of the unordered state.

There is, however, a natural and practical limit to this field expansion, dictated by human sense perception itself. Eventually micro-intervals are encountered which approach the 'subliminal' in pitch discrimination (for example, the Schisma). In addition, complex relations (ratios) are heard as mis-tunings of the simple ratios, which act as powerful NORMS. Traditional scales generally have from five to eight elements, so that much of the potential complexity must be sacrificed in the practical application to a stringed instrument. The wide 'microtonal' field of srutis in India acts as a pool of alternatives for setting up various different scales. In ancient culture, there is no, one, correct scale which dominates the field. It is a culture of intonational pluralism.

The expansion of the matrix must be limited by some rubric to maximize order. Such a rubric is the restriction of numbers to specific families. In 3-Limit monochord sequences, all of the inherent numbers have the form $2^a *3^b$. Musically, they generate a line of fifths-fourths, (a horizontal line on the matrix) which rapidly gains numerosity as it expands. 5-Limit monochord sequences restrict numbers to the form generalized as $2^a *3^b*5^c$. It becomes a triangular matrix which can be derived from the multiplication table of 3*5. Hence any sequence using these numbers as roots can be accurately expressed as a matrix of tuning operations.

In the Egyptian tradition of Heliopolis, the One supreme god (Atum or Temu) symbolizes the first manifestation of vibratory being from the Primeval Waters (Nun). Then from his mouth (also by masturbation, the theme is one of self-manifestation through vibratory power) he generates the great Ogdoad of gods (neters or powers), mirroring the great musical Ogdoad which defines fully extended diatonic harmony. These gods come in reciprocal pairs, as do the harmonies. This metaphor is also used by the Greek Anaximander, for whom the universe is generated as a vortex in reciprocal manner (male and female) from the Apeiron (the infinite field of possibilities). Central to this generation is the necessity for some principle of limit, the concept of Peras. The ancient creation stories were sung by itinerant bards over the whole 'civilized' world, in an aural tradition which valued the sonic experience over the other senses in defining reality.

The sexual metaphor also permeates the Greek theory of Means. Of the three classical Means (Arithmetic, Harmonic, and Geometric), the Geometric Mean is the gateway to temperament and esoteric concerns. The other two means form the metapysical and practical foundation to earlier monochord arithmetic. We see this most clearly in the first two (simplest) 3-Limit monochords. In the series 2:3:4, the number 3 is the Arithmetic Mean between 2 and 4. In the series 3:4:6, the number 4 is the Harmonic Mean between 3 and 6. Note that they are reciprocals to each other. They also associate strongly with the two tuning techniques used for setting a just intonation: the Harmonical Proportion (which uses Harmonic Series componants), and the Arithmetical Proportion (which changes the string length). In either case, the Mean is pictured as the 'offspring' of the bounding pair.

Note that the two sequences 2:3:4 and 3:4:6 are paired or presented together. There is good reason for juxtaposing them in this manner. These two sequences 'exchange places' between the Over Series and the Under Series. Thus they are natural 'partners' or 'friends.' A cursory examination of this material reveals that all of the sequences have a 'partner' of greater numerosity which can be derived from the given division--all except for a small minority which are marked with an asterisk. These harmonies have a special form of symmetry. The rising scale and the falling scale are identical. Thus they combine into unity the two reciprocal monochords that generally relate in a 'yin-

yang' fashion.

The simplest possible such symmetrical monochord is the famous 'Mousike' sequence 6:8:9:12. Note how it combines the two sequences 2:3:4 and 3:4:6. Symmetrical monochords have enhanced metaphysical value because of their inherent balance. In this case, the double 6-12 generates the two tetrachord frames of ancient Hindu-Greek music theory. These tones are 'fixed' in the scale, and form the axis of the scale (the Axis Mundi). The other elements of the scale are 'movable,' in that there are alternatives for the thirds and sixths in the scale. The notion of alternatives is natural to the monochord user who has but to slide the bridge to a new position to establish a new pitch. The tuner must choose a particular scale for the present musical context, and sacrifice the other alternatives. The axis of the scale then fills with more 'descendants.'

The idea of pairing the monochords in this 'partnership' relation came from Ernest McClain's excellant book on ancient cosmology: THE MYTH OF INVARIANCE. His main focus is the partnership between the double 30-60 and the double 72-144. I have generalized the concept to cover all of the cosmological monochords.

There is one more way in which we can employ the sexual metaphor, consistent with the ancient approach. It is summarized in the table showing the 15 monochord 'families' covered by this paper. Take, for example, the number 5. It is the 'parent' of three cosmological monochords: 5-10, 10-20, and 40-80. In other words, those three doubles have as their ultimate origin the number 5. It is amazing just how many of the significant monochords have a common origin. Very prominant are the numbers 3, 9, and 27, which generate many highly important monochord sequences.

Speaking of these primary 3-Limit numbers, they remind us of Plato's Timaeus Lambda, which uses the sequence 1, 3, 9, and 27. If we wish to relate these numbers as harmonics or a monochord sequence, the smallest double we can use to frame them is the sequence 16-32. By finding least common multiples (by the Egyptian method of doubling), 1, which is expressed as 2, becomes 16 and 32, 3 becomes 24 (by doublings), 9 becomes 18, and 27 remains the same. Thus we get the sequence 16:18:24:27:32, which expresses the original sequence within one octave. In effect, this operation brings the numbers into the same 'house,' or makes them 'close kin.' Every monochord will have a member which is not divisible by 2, and which determines the level of numerosity of the sequence. This member is the determinant of the sequence or the 'youngest' member of the family.

The musical significance of each 3-Limit monochord is given with the sequence; but, if I may, I will make some further short commentaries on each division.

The doubles 16-32 and 18-36 are closely related because

they both generate modes of the PYTHAGOREAN TETRAD. Since this harmony has four different elements, there are four possible modes. These can specified as: Ⓒ-G-D-/A, F-Ⓒ-G-D, \Bb-F-Ⓒ-G, and \Eb-\Bb-F-Ⓒ. Every harmony has as many modes as it does elements. Rarely do I give all of the possible modes (they are all in the MONOCHORD NUMBERS REFERENCE MATERIAL), but rather only the most musically significant. The double 16-32 is important since it presents the Tetrad in its smallest possible integers. The sequence 18-36 presents it in its most harmonious position, since it fills the region between comma-lowered and comma-raised dissonances.

We are reminded here of numbers in Egyptian astronomy. They divided the sky into 36 Decans of 10 degrees each, producing the traditional 'star-clock' which was painted on the coffin-lids. The number 36 and its multiple by 10, ie. 360, are persistent cosmological numbers in the measurement of time. Note that the partner of this sequence, the double 24-48 becomes one of the most revered of just diatonic scales when converted into its 5-Limit 'relative.'

The double 48-96 is the smallest integer expression of the ancient and ubiquitous scale popularly known as the Pentatonic Scale. It is also the oldest recorded tuning pattern in existence, if we believe the Chinese historian Sze Ma-chi'en, who wrote (around the era of Ptolemy) that this scale is attributed to LING LUN (court musician to Emperor Huang-ti), who lived in the 27th Century B.C. The reference is interesting

referred to in Plato's Timaeus. If we wish to generate the most harmonious mode of the Pythagorean Hexad, we must use the doubles 144-288 and its compliment 216-432. Again, note the ubiquitous cosmic numbers.

In the double 384-768 we have the smallest integer expression of the 3-Limit diatonic scale. This scale can be placed in symmetrical array, using the double 432-864. In modern terms we would call it a Dorian mode (the classical Greek name was Phrygian). It was one of the two scales that Plato 'allowed' in the republic (the other was 5-Limit double 72-144).

The Pythagorean Ogdoad is given in smallest terms through the double 1152-2304. This is the monochord of the Sectio

these

These graphs show that all 3-Limit harmonies have bi-lateral symmetry. The Axis of Symmetry is shown by the dotted line. Note that the harmonies emphasized by an asterisk have an additional special form of symmetry. The Axis includes the generator. At the opposite end of the Axis is the irrational ratio $\sqrt{2}$, the Geometrical Mean between the octave C-C. Because all 3-Limit harmonies are symmetrical, they form a special subset of the wider expanse of 5-Limit harmony. It must be remembered though, that the vast majority of traditional tunings are 5-Limit; this is the mainstream of tuning tradition. The subtle relations between 3-Limit and 5-Limit harmony are the source of much allegorical lore.

The simplest, primal 5-Limit monochord sequence is the double 3-6. Note that this double already occurred in the 3-Limit group, but now the house admits a 'new brother.' Whereas powers of 3 generate a line of fifths-fourths (the interval of power), the number 5 generates the intervals of beauty, the musical thirds and sixths. In this monochord we have 'birthed' the three Axes of Just Intonation, which defines the three tuning procedures (namely fifths, major thirds, and minor thirds). The diagonal lines on the matrix indicate tuning in just thirds.

The sequence 3-6 is called the Senarius series, the emergence of the just major and minor triads, foundation structures of music. When this sequence is multiplied by 10 to give 30-60, we have the monochord closely associated with Babylonian mythology. The first 60 numbers are gods. The principal gods are 10(Adad), 20(Shamash), 30(Sin), 40(Ea-Enki), 50(Bel-Enlil), and 60(Anu-An). Of these, the greatest are 40, 50, and 60. Since these numbers are often related to the reciprocal table which uses 60 as its 'transformation point,' we can associate these numbers with 40:60=2:3, 50:60=5:6, and 60:60=1:1. They also relate as 4:5:6 so that they emphasize the three sacred roots 2, 3, and 5. Finally, they can also be assigned to the three Axes of Just tuning on the matrix.

Note that many of the ancient religions had a trinity of greatest gods, reflecting the three sacred roots. We even see the ghost of this trinity in Christianity. These sacred roots (which are also the sacred roots of geometry), are all aspects or 'masks' of the One, the ever-present generator and source of all.

The monochord 3-6 is the first 'double-strand' tuning on the matrix. A quick glance at the family of 5-Limit monochords reveals that there are two main themes: double-strand patterns (a double line of fifths) and triple-strand patterns (a triple line of fifths). We will touch on the quadruple-strand only briefly. The core double-strand harmonies are the diatonic scales, three modes given in 24-48, 30-60, and 36-72. Triple-strand harmonies are call chromatic, for example the sequences 360-720 or the even more expanded harmony 4320-8640. The core chromatic pattern is the sequence 60-120.

The next monochord of interest after 3-6 is 4-8. It

presents the musical triads in a particularly harmonious mode. It is also last in the series of metaphysical monochords that generate the four levels of Pythagorean harmony (related to the four levels of the tetractys, and to the four Hindu castes and the four classes in Plato's societies). These four monochords are 1-2, 2-4, 3-6, and 4-8. The sequence 1-2 generates the one 2-Limit harmony. The sequence 2:3:4 generates the universe of 3-Limit harmony and the line of fifths. The sequence 3:4:5:6 gives birth to 5-Limit harmony and the triangular grid-matrix. Finally, the sequence 4:5:6:7:8 gives birth to 7-Limit harmony which (by the extension of our model) is tetrahedral in its matrix. These are all of the primes less than 10. Although I am not covering 7-Limit harmony here, it is interesting that the first four monochords generate the four possible levels of complexity in the rubric of our scales.

You may have noticed that there are a lot more 5-Limit harmonies of consequence than 3-Limit, but that the numerosity of the patterns does not grow as rapidly. If we were to consider 7-Limit monochords, the same progression ensues--more monochords, smaller numbers. The family of 7-Limit monochords deserves a paper on its own. Higher primes like 11, 13 or 17 create unworkable complexity; they are quite 'exotic.' Nevertheless, the Egyptian-Greek scientist Ptolemy notated the beautiful (when you get used to it!) tetrachord 9:10:11:12. Such a monochord could only have been considered when we remember the long history of intonational sophistication bequeathed by Pythagorean scientists. Ptolemy was also one of the earliest writers to specify significant double-strand tunings. Hence, in my other writings, I have often labelled double-strand patterns as PTOLEMAIC TUNINGS.

The next extension of the double-strand in the matrix is the sequence 5-10, called the Tetractys series. Here are birthed the two wholesteps which are the hall-mark of just tuning. The ratios 8:9 and 9:10 define two wholetones which are a Syntonic Comma apart in size. The Syntonic Comma is one of the controlling micro-intervals which typifies just intonation. A look at the large circular graph after the monochords reveals that it is convenient to view the entire field of just tuning as a matrix of commas. There is an old Pythagorean tradition (from the fragments of Philolaus) that the 8:9 wholetone is divisible into 9 commas. The 9:10 ratio is 8 commas in size. The various sizes of just semi-tones are also conveniently described in comma size; indeed, any practical musical ratio on our grid can be classified that way. We shall see later that a second micro-interval must be specified in the notation, the schisma. With these two new signs all of the material in our investigation can be specified without ambiguity.

We do not need to consider the schisma as yet, but the comma must be accepted. We saw the Syntonic Comma already in the 3-Limit monochords. Any natural pitch also has a comma-raised and a comma-lowered dissonance in adjoining regions of the matrix. The comma is an interesting interval. It is too small to use as a scalar element, (its ratio is 80:81), yet it is too large to be

ignored. One must make a choice between scalar elements that relate by a comma. In India, the comma is called the SRUTI in the Nāṭyashāstra. According to tradition, there are three sizes of srutis, shown underneath the large circular graph. The Pramana Sruti is one comma in size, the other two are 3 commas and 4 commas. It is significant that these three intervals define the boundaries of the fully expanded triple-strand which is the field of srutis. This is one of the reasons for concluding that the triple-strand is the ancient Hindu-Babylonian tuning system.

The double 5-10 is also called the Tetractys series because the ratios formed by the first 10 integers have the greatest power as NORMS of harmony. This series of ratios ends with 8:9 and 9:10. The number 10 is often evident in the relations between our monochords. So many of them relate by 10 that I listed all 15 groups in a small graph under the graph of the families. It is significant that, if we want to express all ten ratios (including those of 7) as a monochord sequence, the smallest expression is the 7-Limit double 2520-5040. Plato gives this monochord in the allegory of Magnesia, as the number of 'citizens' in the city. Various Platonic 'cities' are also monochords.

The double 6-12 is another mode of 5-10. This time it is in its most harmonious position, as if ready to expand into the Ptolemaic Pentad 12-24. Note that the 3-Limit monochord 6-12 implied the 10, even though it was not included. The element of 5 is the 'playful' element in the expansion. The god Bel is the god who sits on the mountain and unleashes the 'flood.' Note that the partner of 6-12 is 30-60, the Babylonian series.

The harmony which we call the Ptolemaic Pentad is presented in smallest integers as 8-16, and in two more modes as 10-20 and 12-24. This is the first, truly manifested double-strand harmony, with two lines of fifths connected by just thirds. Looking at the sequence 8-16, which belongs to an illustrious family of monochords (whose ultimate parent is 2), we see in it two significant harmonies. We have the CM7 chord in pitches C-E-G-B, and the Em7 chord in pitches E-G-B-D. These are the two Ptolemaic Tetrads, musically important subsets of the Pentad. We have already seen the sequence 10-20 as the partner of 3-6. Notice how it is a natural extension of 5-10. It is also the first expression of the diatonic tetrachord 15:16:18:20. Finally, the mode 12-24 is the most balanced and harmonious position of the harmony. None of the 5-Limit monochords yet covered are symmetrical. You may note that there are comparatively fewer symmetrical monochords in the 5-Limit group as opposed to the 3-Limit group (tribe?). Consequently, symmetrical 5-Limit monochords are even more precious than 3-Limit ones.

Some 5-Limit harmonies have bi-lateral symmetry. A small number have the special symmetry about the generator that we already saw in some 3-Limit harmonies. Obviously these have enhanced importance.

We have already seen the sequence 15-30 as the compliment or partner of 4-8. But it also has the distinction of being the first and simplest chromatic (triple-strand) harmony. Thus it embodies the small chromatic semitone (ratio 24:25 or 3 commas in size), as well as the large diatonic semitone (ratio 15:16 or 5 commas in size). Interestingly enough, it also contains the over-sized semitone whose ratio is 25:27 (6 commas in size). In highlighting these alternative semitones, this sequence shows the tendancy of just intonation to yield interval complexity. Although all of these ratios are heard as semitones, they have unique sizes and characters. Just intonation is prone to the generation of this microtonal variety. Hence a more exacting notation is necessary than that used for standard 12-Equal-temperament notation.

The compliment of 15-30 is the seminal chromatic sequence 360-720. This series presents the essential chromatic scale (minus the tritone) within a symmetrical array. We will have more of chromatic harmony later. Let us focus here on the double-strand expansion.

The series 24-48 (the two PTOLEMAIC HEPTADS) is one of the most revered of practical and harmonious diatonic scale tunings, since it presents the 5-Limit scale in its smallest integers. At the same time it yields very significant modes of the pattern. The Over series is the major (Ionian) scale in its most consonant form. Its reciprocal pair on the monochord is an important Phrygian minor scale. Looking at the ratios of the Over series, some important intervals are birthed. The just tritone

the 'official' year is very close to a comma in size. It is also possible to divide the 360 day 'year' into 72 weeks of 5 days each, or 60 weeks of 6 days. Even the division of the 365 day year into 7 day weeks yields a musical association, for then we have 52+ weeks in the year, an allusion to the 53 commas in an octave. It appears that the ancients did their best to make the calendar 'fit' the musically important monochord numbers. Of course, the astronomical numbers always have a 'messy' or 'not quite commensurate' quality about them. The actual lunar cycle is a little less than the desired 30 day period, the year a little more than 360 days. The mathematics associated with musical concerns is far less complex than that of accurate calendrics. I would assume that Canonics was an older science to be actualized. The challenge of accurate astronomy then inspired a more complex mathematics to evolve in the early civilizations, especially Sumer-Babylonia.

Note that the sequence 30-60 yields an Ionian and a Phyrgian scale which is different from 24-48. Again we see the establishment of alternative NORMS in the scale. Even though these two scales are almost the same, the corresponding mood or quality of each harmony is noticably different. Hindu music recognizes potentially hundreds of alternative modes of the diatonic scale. Of course, factors other than intonation also come into play when deciding the classification of a Raga.

The series 36-72 yields a harmonious Mixolydian mode and its reciprocal, the natural (Aeolian) minor scale. There are several (four) more modes of this harmony which have not been considered here. The three covered are the most harmonious of the seven. The Ptolemaic Heptads are among the most revered of traditional harmonies, second (possibly) only to the Ptolemaic Ogdoad.

The next sequence in numerical order is 40-80, but we will continue to examine the double-strand before commenting on the triple-strand. The next double-strand tuning in the series is one of the most important from a structural viewpoint: the double 72-144. It is the PTOLEMAIC ENNEAD, whose compliment is again the double 360-720. This Ennead has as subsets the two PTOLEMAIC OGDOADS shown in abstract in the middle of the page. The right-hand one is the harmony which 'fills up' the double-strand region. In other words, any further double-strand expansion results in elements which are comma-related. We see this happening by observing the abstract Ogdoad on the left, where the harmony embodies both D and \D. The Ptolemaic Ogdoad is thus the furthest expanse of diatonic harmony and a major structural milestone in the just harmony matrix. It also embodies the 5-Limit medial semitone of ratio 128:135 (about 4 commas in size). We shall say more about this semitone later.

Although the Ptolemaic Ennead is not symmetrical, the two Ptolemaic Ogdoads which is in its 'house' have bi-lateral symmetry. In fact, all of the even-numbered Ptolemaic patterns have this symmetry. Odd-numbered Ptolemaic patterns have a yin-

yang pair characteristic of harmonies that are not symmetrical. Two more modes of the Ennead are presented as 90-180 and 96-192. 90-180 has the Under series in its most harmonious position. The comma-related alternatives are emphasized by using a bracket. The series 96-192 is the series implied by Plato in the Timaeus construction. Here the Over series is in its most harmonious position. Again the comma-alternative is bracketed.

The double 90-180 also has a prominant ancient Greek scale which is technically chromatic (triple-strand), but is Dorian in character. It is the first pattern of consequence after the series 60-120 to have special symmetry. Note that this Dorian scale is different in character from the 3-Limit Dorian scale of sequence 432-864. The ancient Greeks claimed that they derived their scales from the East (Babylonia-Anatolia), but they could also have inherited them from the South (Egypt).

At the bottom of the page we see, in the abstract, what happens when we set up a harmony which is a very wide double-strand expansion. We encounter a pitch which is but a SCHISMA sharp of the generator. The schisma lies at the edge of our ability for pitch discrimination, and far beyond what could be used as an element in the scale. It is also at the limit of practicality in bridge calibration on the monochord. As a ratio the schisma embodies the numbers 32,768:32,805; or we could express it in prime factors as $2^{15}:3^{8}*5$. It is encountered whenever one attempts a very wide horizontal expansion of the matrix. Needless to say, it is pointless to expand the Ptolemaic patterns beyond these bounds. The notation graph given at the end follows the 3-Limit line of fifths to the establishment of the PYTHAGOREAN (or DITONIC) COMMA, which has the enormous ratio of 524,288:531,441 (in prime factors $2^{19}:3^{12}$). It is none other than the Ptolemaic Comma (11 schismas in size) raised one schisma to give a comma which is 12 schismas in size.

One more pair of double-strand patterns is given, the series marked 216-432. Such wide double-strand expansions have alternative pitches which relate by a Syntonic comma. Hence they form a 'clan' of pitches, out of which one can choose elements for the scale. There is historical record of the wide double-strand acting in such a way as an 'extended family.' The old Arabic and Persian scale system (9th Century Baghdad) defined 17 potential tones in the octave. In fact, it is the wide pattern given as the maximum expansion, but leaving out the schismatic pitch at the end. Our double 216-432 is 'on the way' to this complexity; it is an 'ancestor' of lower numerosity. This harmony is called the Ptolemaic Undecad (it has 11 elements). We will not consider further expansions to 13, 15, and 17 elements here. Perhaps the reader would like to compute the relevant numbers as an exercise in musical arithmetic.

With this pattern we leave double-strand harmonies and look at the triple-strand. We saw the first chromatic pattern in the series 15-30, but the truly manifested triple-strand core pattern emerges in the series 40-80, 45-90, and 60-120. In

40-80 we have the smallest possible integers, and its compliment 45-90 has the distinction of being the next entry in the complete compendium of possible 5-Limit monochords. In addition, it has the distinction of being the first monochord to admit the Syntonic comma ratio 80:81.

The sequence 60-120 has the distinction of presenting the harmony in an array with special symmetry about the Tonic.

The importance of this monochord to the structure of just harmony cannot be over-emphasized. It is the first symmetrical statement of the 5-Limit musical consonances of music. That is, the inherent intervals C-C, C-F and C-G are the Perfect Consonances, C-A and C-E are the Medial Consonances, and finally C-Eb and C-Ab are the so-called Imperfect Consonances. The six tones F, G, A, E, Eb, and Ab are the closest possible harmonic relations to the generator tone C. This pattern defines the SIX TUNING DIRECTIONS of (5-Limit) just intonation. (The 3-Limit subset has only two tuning directions, west or east).This hexagon pattern exhausts all of the possible musical consonances (if we avoid the controvercial issue of the consonance of the 7-Limit ratios 4:7, 5:7, and 6:7; ratios associated with esoteric Pythagorean tunings). Ellis in the 19th Century called it the HARMONIC HEPTAD. We would expect to see these numbers as prominent in cosmology, and we are not dissappointed. The Babylonians used base 60 in their astronomy. Note that the prestigious number 60 equals 2^2*3*5. It is not generally appreciated just what a convenient number base this is for musical arithmetic. It also allows easy conversion from base 10, base 12, and base 20. The monochord 60-120 was a standard one in the ancient world. Ptolemy used it, expressing ratios that are not integral subsets as sexagesimal fractions (like our degrees, minutes and seconds). Thus he was happy to consider numbers to the order of 60^3, or 216,000. He forms an important link at the end of the Greek world between Greek harmonics and the older traditions.

The sequence 64-128 is included because it forms the first and simplest division to exhibit the quadruple-strand. It also forms the Tetractys pattern on the matrix. Although not the main focus of this paper, we should have a little closer look at the quadruple-strand patterns. The key interval here is the DIESIS or ENHARMONIC, of ratio 125:128. A Diesis is the difference between a sharp and a flat. A glance at the large circular graph shows it to be two commas in size, for example, between pitches C# and Db. However, in order to clarify the deep structure of just harmony, we must take into account here the reality of the schisma.

The schisma is like a musical smallest unit or atom. All larger intervals can be described with good accuracy in schisma units. Most notably as mentioned above, the Syntonic Comma is 11 schismas in size. This is the interval between C# and /C#. Now the Diesis C#-Db is 21 schismas in size. Therefore the 'comma' /C#-Db is 10 schismas in size. You can see why we must use the term 'comma' loosely at times, since there is actually

more than one size of comma. In fact, we have already seen even another size of comma, the Ditonic (12 schismas). Even though they are slightly different in size and numerosity, we hear all of them as commas, since we are dealing with such a fine resolution of the field. The 10 schisma comma is called by moderns the DIASCHISMA of ratio 2025:2048. Now since we saw that 21=11+10, it is also true that 21=10+11. Thus there is another ratio which we could have used in place of /C# that is practically identical--only a schisma flatter. It is \Db.

/C# is ratio 128:135. The 3-Limit ratio 243:256 (called the Pythagorean semitone or Leimma) is only a schisma flat of it. Just as the ratios 8:9 and 9:10 were the smallest and most balanced ratios to generate the Syntonic comma (and create the COMMA BOUNDARY in the regionalism of the just matrix), in the same way the ratios given above are the smallest and most balanced way of generating the schisma and the SCHISMA BOUNDARY. Looking at the reference graph for the just field at the end of the diagrams, we see why the pitches that straddle the schisma boundary (the solid heavy lines) must have two pitch names to be unambiguous. Thus the ratio 128:135 is notated as /C# and \Db; whereas, ratio 243:256 is notated as /C# and \Db. In other words, pitches which sit beyond the schisma boundary are schisma altered forms of the pitches within the central zone. An accurate notation for just intonation must clarify these relations.

However, when using a practical comma resolution of the scale, we still think of the Diesis C#-Db as two commas in size. We have the choice of what level of resolution we wish for our field. The comma is practical and easy to erect on the monochord. The schisma is more accurate but theoretical. By contrast, the modern western approach using temperament employs a comparatively shallow resolution to the medial semitone, eliminating the schisma, comma and the diesis. Just intonation demands at least the comma and diesis. There are 53 commas in the octave (most are 11 schismas in size, some are 10, some are 12). This fact draws us to a 'compromise' scale in which the commas are 'normalized' (ie. made identical in size, about 11.5 schismas); but here we are approaching temperament.

The sequence 120-240 presents the next expansion of the triple-strand. Its compliment 180-360 is also implicated. Within these two doubles can also be found the smallest integers for two core chromatic harmonies (two Ennead harmonies which form the core of chromatic scales). These structures are also symmetrical about the generator. The sequence 120-240 also yields the chromatic interval called the just augmented sixth between Db-B, which has ratio 128:225. This interval is very close in size to the primal 7-Limit ratio 4:7, the relationship between them being the important micro-interval called the SEPTIMAL CLISMA of ratio 224:225 (4 schismas in size). This ratio dominates the relations between 5-Limit and 7-Limit harmonies.

The symmetrical sequence 360-720 provides the next expansion of the chromatic harmonies. Note that the two tritones

The Nāṭyashāstra says that the octave is divisible into 22 srutis. Well, we see 25 here. This book is filled with a number of such 'errors,' which appear to be deliberate. It is obvious that the sruti is the comma, but the references to structure are often given in such a way that one who has not done his homework with monochord mathematics will be mis-led. In this case, it can be shown that there are exactly 22 commas in the musical fourth (the ancient tetrachord frame). Meanwhile, Indian music culture later abandoned the monochord altogether. They did not really need it, since the intervals are so well entrenched 'in the ear,' and traditions are passed on directly from teacher to student. There is no reason to believe that modern musicians tune any different from the ancient practice--that is, until the introduction of western harmoniums and keyboards exposed them to western musical practice.

The situation now is much different than the 19th and 20th Century where one tuning ideal formed an 'imperial' norm for music in general. Modern electronic keyboards are easily retunable, and many musicians are now interested in using Asian intonations, just as many Asians are using 12-Equal-temperament. In fact, the possibilities now are greatly expanded. Harry Partch used the 11-Limit beyond the 7-Limit restriction voluntarily imposed by the Pythagoreans. The world of musical intonation is now open to vast possibilities, due to a much more comprehensive understanding of resonance systems in general (tempered and untempered). It will take a while for this essentially mathematical information to be 'absorbed' by the bulk of modern musicians, many of whom are quite 'illiterate' in regard to the scientific basis of their art. Nevertheless, there is a growing awareness.

As a practical means of learning the foundations of musical acoustics, the monochord is still recommended. If a person sets up the tunings in this list he will go far in experiencing the norms of just harmony. Thus the monochord has much to teach us today, as well as providing a key to ancient cosmology. The early science of cosmology was closely tied to musical concerns; it may have evolved directly out of the earlier science of harmonics. It was a cosmology which put the structure of the aural-sensorium in central place as the generator of norms (laws). The patterns of harmony formed the metaphysical base for an experience of 'reality' that transcends the illusion of visual fragmentation. Temporal phenomena are a better guide to the inner nature of things. Form is inconstant, but the 1:2 ratio is an invariant sonic norm (the octave). Thus the proportional laws of sound form the epistemological base for ancient philosophy and science.

I have tried in this paper to focus on the purely musical attributes of these monochords, showing why they have importance from the stand-point of musical structure. Even though I have been a brief as possible, I have also introduced some of the most important just musical ratios, and the monochord

sequences that gave them birth. Lastly, I have extended the allegory of family relations for these procedures, and shown why this description is appropriate to the relevant mathematics. It is my hope that this list of monochords will act as worth-while reference material for the examination of harmonics in ancient cultures.