Means to Music

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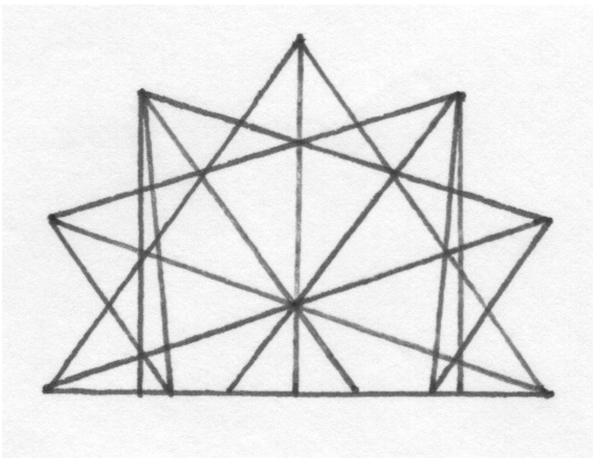

Siemen Terpstra

www.siementerpstra.com

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THE GENERATION OF THE MUSICAL RATIOS THROUGH THE APPLICATION OF MEANS

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We usually understand the ratios of JI as derived from the Harmonic Series. Any Just interval can be viewed as a frequency ratio between two whole numbers, numbers which refer directly to the Series itself. However, it may not be generally appreciated that there is also an alternative way to generate the Just ratios. This method has great historical importance as well as being of intrinsic interest in itself. I refer to the procedure of examining the mean between two extremes.

I will side-step most historical considerations for this short article. Suffice it to say that the ancient Babylonians, Egyptians, Greeks, and Chinese all tended to look at life as a dynamically balanced "middle path" between opposites. The Pythagorean Greek School of philosophy, being number based, must have placed great emphasis on the discovery of an orderly and finite mean between two "boundary" numbers. The mean becomes the 'offspring' or direct descendent of the two generating 'parents.' It shares in the absolute qualities of its bounding and life-giving pair of extremes.

We will use the three 'classical' means for our procedures. These are the Geometrical Mean, the Arithmetic Mean, and the Harmonic Mean. The writings of Archytas and Plato refer to these three alone. The set was expanded to six by Eudoxus, and eventually ten by Pappus and Eratosthenes. The three primary means are all we need to generate the ratios of music.

It is interesting that the Arithmetic Mean and the Harmonic Mean always generate whole-numbered ratios. It is chiefly through these two means that the intervals are 'birthed.' The Geometrical Mean, on the other hand, always generates irrational ratios, since it is found through the square root of the product of the extremes. Such irrational ratios are not associated with JI, rather with systems of temperament. They do not have practical applications, but they do have esoteric significance for the Pythagorean School.

We may summerize the formula and solution to the three means through simple modern algebra. Two ratios are given, call them a and c, and their mean b. For the Geometric Mean, the formula is $a/b=b/c$. Thus the solution for the mean is $b=\sqrt{ac}$. The formula for the Arithmetic Mean is $a-b=b-c$. Consequently, the solution is $b=a+c/2$. Finally, the formula for the Harmonic Mean is $a-b/a=b-c/c$. Thus the solution for the mean is $b=2ac/a+c$.

These three means have intriguing relations to each other. For example, the Geometic Mean is itself the mean between two opposing means, the Arithmetic Mean and the Harmonic Mean. These relations will become clear when we apply the procedure.

We begin our investigation by observing the means between the Unison (ratio 1/1) and the Octave (ratio 2/1). The Harmonic Mean is the Just musical Fifth (ratio 3/2), and the Arithmetic Mean is the Just musical Fourth (ratio 4/3). The Geometrical Mean is the ratio $\sqrt{2}$, which we associate with the 12-Equal-Tempered Tritone. It lies midway between the other two means, the relation referred to in the previous paragraph. If the tone C is our starting point, then the means refer to the pitches F and G. These three tones form the tetrachord frames of ancient music theory, and are still revered in modern theory as the dominant-subdominant axis. The fact that the musical consonants of the Fifth and Fourth could be generated through the Arithmetic Mean and Harmonic Mean between the Unison and the Octave must have greatly pleased the ancient Greek mind.

Let us continue our investigation by examining the means between 1/1 and 3/2, the musical Fifth. The Harmonic Mean is 5/4, the Just musical Major Third. The Arithmetic Mean is 6/5, the Just musical Minor Third. The Geometrical Mean is the ratio $\sqrt{6}/2$, which is irrational and could be called a 'neutral' Third, since it sits midway between the two Just Thirds. We will not use this one. Is it not wonderful that the means between the 'absolute' consonance of the Fifth should be the 'medial' consonances of the musical Thirds?

Let us look at the means between another Fifth, say F (ratio 4/3) and C (ratio 2/1). The Harmonic Mean is the Just Major Sixth (ratio 5/3) or pitch A, and the Arithmetic Mean is the the Just Minor Sixth (ratio 8/5) or pitch Ab. The Geometric Mean again forms the 'neutral' tone between, at the irrational ratio $4/\sqrt{6}$. Thus we can conclude that the means between any musical Fifth results in a composite structure which defines the Just Major Triad and the Just Minor Triad.

We can also make another conclusion from our investigation thus far. This inference is best visualized with the aid of the accompanying diagrams. Take the Fourth and Fifth between the Unison and Octave. The tones considered as a set form a symmetrical pattern and hence have an axis of symmetry. This axis passes through the Geometrical Mean. Similarly, the means between the musical Fifth C-G form a symmetrical pattern whose axis of symmetry falls through the Geometric Mean. See the included diagrams to clarify this relation. The circular model converts the octave to a round of 360 degrees, so that any ratio can be expressed as a discrete portion of a round. Such a conversion is the result of a simple logarithmic transformation. These diagrams render the axis of symmetry clear and visible.

We continue our investigation by finding the means between the 1/1 and the 4/3 (musical Fourth). Now the Harmonic Mean is the Septimal Sub-minor Third (ratio 7/6), and the Arithmetic Mean is the Septimal Wholestep (ratio 8/7). The Geometric Mean is

$2/\sqrt{3}$--a ratio of little interest to us. Note here that the means between the Octave generate the principal 3-Limit ratios; then the means between the Fifth generate the principal 5-Limit ratios. Now the means between the Fourth generates the principal 7-Limit ratios. As another example, look at the means between the Fourth G (ratio 3/2) and C (ratio 2/1). The Harmonic Mean is 7/4 (the Septimal Minor Seventh) and the Arithmetic Mean is 12/7 (the Septimal Major Sixth). The Geometric Mean is $\sqrt{3}$.

It seems logical to examine the means between the Just Major Third (ratios 1/1 and 5/4) next. We now generate the Just wholesteps. The Harmonic Mean is the ratio 9/8 (Just Major Wholetone), and the Arithmetic Mean is the ratio 10/9 (Just Minor Wholetone). These two ratios are separated by the Syntonic Comma which is an ever present feature of JI. We could also say that we have generated the '9-Limit,' although 9, being a composite number, should rightly be considered as a subset of the 3-Limit.

Of particular interest here is the Geometric Mean of the Major Third. It is the ratio $\sqrt{5}/2$--the meantone which is the basis of the One-Quarter-Comma Meantone Temperament system (and hence also its close variant, 31-Equal-Temperament). Here we finally have a Geometric Mean which is musically useful. But let us adhere to JI and not deviate into tempered systems.

Another example of the means between the Major Third is the case of the Third Ab-C (ratios 8/5 and 2/1). The means are the alternative Just Sevenths of ratio 9/5 and 16/9. Hence, the means between any Just Major Third results in two tones which relate by a Syntonic Comma.

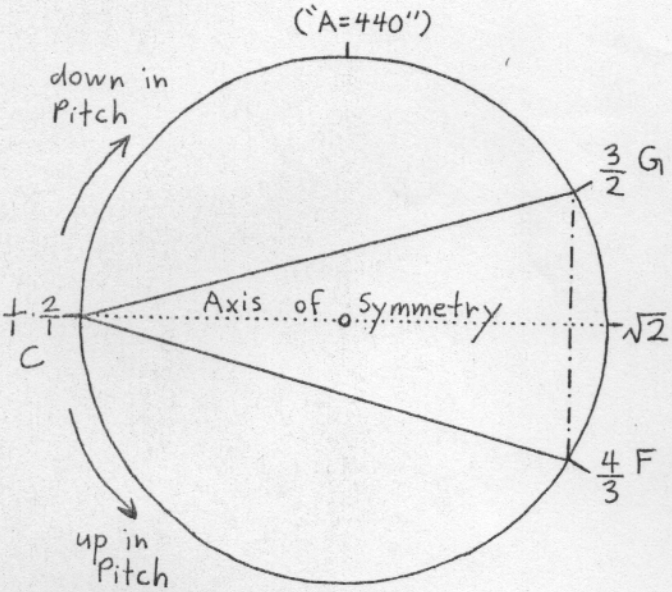
It is logical to examine the means between the Minor Third (ratios 1/1 and 6/5) next in our progression. Now we get 11-Limit ratios, 11/10 for the Harmonic ratio and 12/11 for the Arithmetic Mean. We are moving into more unfamiliar musical territory. We could press on to find the means between unison and the Sub-minor Third (ratios 1/1 and 7/6). The Harmonic Mean is 13/12, and the Arithmetic Mean is 14/13. We are now generating 13-Limit ratios. Shall we carry on? The means between Unison and the Septimal Wholetone (ratio 8/7) are the ratios 15/14 and 16/15. The latter ratio is the Just Diatonic Semitone. Let us take these operations but one more step. The means between Unison and the Major Wholetone (ratios 1/1 and 9/8) are ratios 17/16 and 18/17. The 17-Limit is now generated. Have we gone far enough? I think so, although the procedures can be continued to whatever limit we choose.

If you have followed these procedures carefully, you may have noticed a simple rule for finding the means between Unison and any given Superparticular ratio. Just double the given numbers and insert the missing one. For example, the means between Unison and the Minor Third (ratio 6/5) are 12/11 and 11/10. These ratios come from the series 12:11:10. The Harmonic Mean is always the larger ratio (ie. the ratio with smaller frequency numbers).

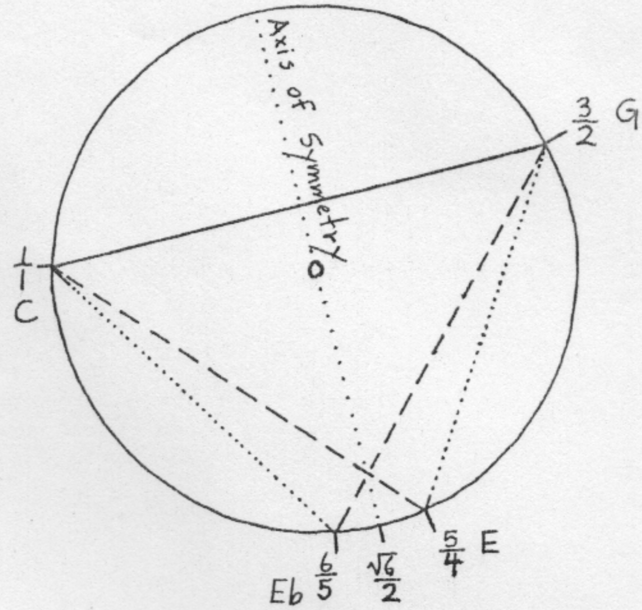
These simple procedures generate all the important ratios of JI music in an orderly manner. Although we have used frequency ratios in our calculations, we could just as well have used string lengths on a monochord (since frequency and string length are inversely proportional). Hence this information was surely clear to the ancient Greeks, as seen through the writings of Archytas. This theory of means, along with a small number of important irrational means (principally the Golden Section), had important applications for sculpture, art, and architecture as well as music. Hence it was the basis of a general theory of Aesthetics, especially in the Platonic School. I hope that this article clarifies this little-understood relation between music, mathematics, and philosophy.

<u>Means and Musical Intervals</u> – examples on the Circular Graph:

A full round of 360° gives an Octave.

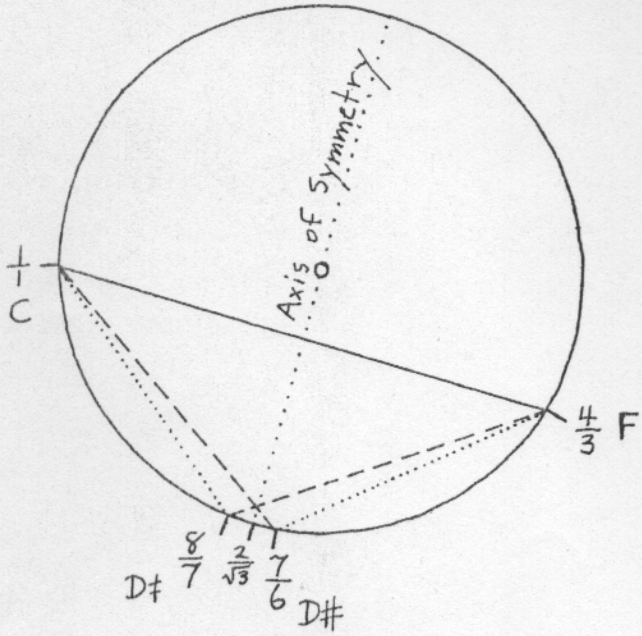


The means between the Octave (Pitches C–C) generate the primary 3-Limit ratios.



The means between the Fifth (Pitches C–G) showing how the Geometrical Mean forms a "neutral" Third.

The means between the Fourth (C–F) generate Septimal ratios.



The means between the Major Third (C–E) generate the Just Whole tones and also the Meantone ratio √5/2.