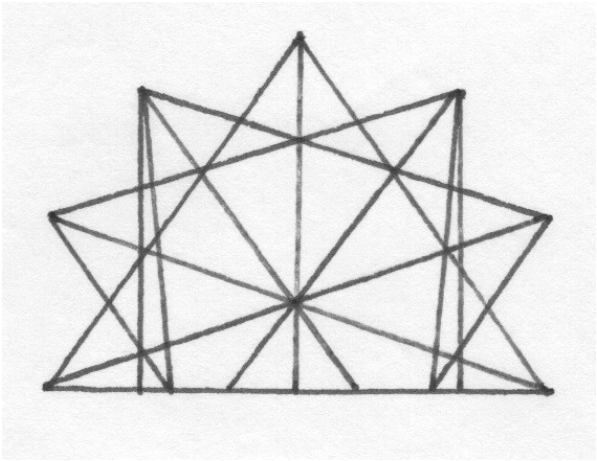
The Meaning of the Tetractys

Musical Symbolism in Pythagorean Arithmology



Siemen Terpstra

www.siementerpstra.com

THE MEANING OF THE TETRACTYS
MUSICAL SYMBOLISM IN PYTHAGOREAN ARITHMOLOGY

SIEMEN TERPSTRA

The Tetractys symbol is an exquisite example of how a simple visual pattern can have multi-varied meanings and 'levels' of interpretive significance. The famous Pythagorean oath refers to the Tetractys, praising it as the source "which contains the fount and root of eternal nature." A profusion of insights can be derived from the image, concepts which are relevant to arithmetic (number relations), harmonics (musical tuning theory), and geometry (number relations in space). Only a pitifully small amount of ancient commentary has been preserved; but the study of the above disciplines uncovers more and more relevance for the symbol. In this short paper, I will briefly review some historical interpretations and offer a few insights of my own that come from extensive work in harmonics and sacred geometry.

The Pythagoreans saw in Number the eternal archetypes of order in nature, the mind, and the metaphysical realm. Hence statements of simple arithmetical relations are imbued with religious import. Mathematics is still the most abstract language of mankind, and the best suited to describe the laws which govern the universe. Thus the practice of mathematical inquiry became for the Pythagoreans a 'yoga' through which one approaches the divine.

There is a long history of arithmology in the Greek world and other ancient cultures. The practice of the various sciences cannot easily be separated from this number mysticism. This reverant attitude to number and pattern also led to the development of Gematria, since the Greeks used the letters of their alphabet to signify the numbers. Even though Gematria is a distinct practice from harmonics and geometry, there are various ways in which concepts and patterns overlap. Long after the demise of the Pythagorean schools, the attitudes associated with these philosophers lived on among the Jewish Kabbalists (eg. Philo), and among Christians (eg. Clement of Alexandria). Among Greeks, there is relevant information from writers spanning a thousand years from the 5th Century B.C. onwards until the extinction of the ancient sciences by Christian orthodoxy.

I cannot possibly cover the whole field of arithmology in such a short format as this. The focus of this paper is the Tetractys itself and the number relations which can be derived from it.

The Tetractys is the product of an emanationist cosmology, in which the first 10 numbers are archetypes of order derived from Unity, expressing essential qualities inherent in Unity, and simultaneously present within Unity. It is these qualities that the Divine Artificer uses to adorn the first principles of creation. The term cosmos means the universe 'adorned' with order. Numbers are the primary tools for the

Square Numbers maintain their pattern. Square numbers are thus associated with Limit or the unchanging. They are also sometimes called 'equal-times-equal'. Oblong numbers are called 'heteromecic,' which means that the factors of their sides differ by one. For example, 6=2*3, 12=3*4, 20=4*5, and so on.

The Greeks not only used a triangular and a square grid to see the spacial properties of numbers, they also formed arrays called POLYGONAL NUMBERS, in which the gnomon can no longer be square. I will not explore these patterns in this paper, since the tradition holds that only the square and the triangle are needed to generate the 'universe.' The only polygonal array to be mentioned here is the Pentagonal Number series, which begins 1, 5, 12, 22, 35... It can be generated by the gnomon expressed generally as 1+4+7+...+(3n-2)=n(3n-1)/2.

The Greeks found many ways to classify the properties of numbers. A LINEAR NUMBER can be pictured as a straight line of dots. Any number can be linear, but only a PRIME NUMBER is exclusively linear. A PLANE NUMBER can be expressed as an array of dots in a plane figure. Thus the Triangular Numbers, Square Numbers, and Oblong Numbers are all planar. A SOLID NUMBER can be displayed as a three-dimensional array of dots. Such a number is either CUBICAL, like 8, which is 2^3, or SCALENE, which is the product of three unequal numbers.

Some classifications of numbers come straight from geometry. For example, a SIDE NUMBER is paired with a DIAGONAL NUMBER when measuring squares or rectangles. Other classifications come from arithmetical properties in themselves. For example, a PERFECT NUMBER (or 'complete' number) is equal to the sum of its factors; for example, 6=1*2*3 and 6=1+2+3. On the other hand, a DEFECTIVE NUMBER (or Imperfect Number) is greater than the sum of its factors; for example, 8=1*2*4 but these factors add up to 7.

Finally, some of the properties evolve out of harmonic considerations. For example, numbers are classified into two groups: PRIME and COMPOSITE (or Secondary). A prime number can only be factored by 1 and itself. Composite numbers are numbers which are not prime. The science of Harmonics is precisely an exploration of numbers as ratios; that is, how numbers relate to each other through division-multiplication. At the very core of this study is the 1ST LAW OF ARITHMETIC: every number is either prime or composite; any composite number can be uniquely expressed (or decomposed) as a product of primes. For example, the number 7 is prime because its only factors are 1 and 7; but the number 60 is composite. It can be expressed uniquely as a product of prime numbers. $60=2^2*3*5$. We shall see that there are enormous implications of this law for the study of harmony.

Some additive properties of numbers were undoubtably known by the ancients, yet are not given in the Greek fragments. The most significant Tetractys-related pattern is known as Mt. Meru or Mt. Sumeru (which gives a clue to the probable origin of

the pattern). It appears in a Hindu text by Piṇgala (c. 200 B.C.), and is shown as Figure 4. On the left we see the traditional expression of 'the holy mountain.' On the right, we see the abstract, algebraic expression which is now called the Binomial Theorem or Pascal's Triangle. Whereas the Tetractys in its classical form is entirely abstract, here is the simplest concrete numerical application of the pattern. Each number is found by adding the two numbers above it to the right and left. Like the Triangular Number series pattern, it can be expanded indefinitely.

This pattern has a number of remarkable properties. For example, the addition of any horizontal line always yields some power of 2. In addition, one can find within it the natural sequence 1, 2, 3, 4, 5... and also the Triangular Number sequence 1, 3, 6, 10... Thus there are close affinities to the other numerical sequences associated with the Tetractys given above.

The name reminds us that all of the ancient cultures had the notion of a sacred mountain at the centre of the world, the spot from where the rest of the cosmos was generated. The Greeks had Mt. Olympus and Mt. Parnassus at Delphi. The Hindus and Buddhists had Mt. Meru and Mt. Kailasa. The Babylonians (after the Sumerians) built temples (ziggurats) shaped like this diagram, to stand for the holy mountain where the gods interact with humans. The Sumerians had the twin peaked Mt. Mashu, from which flows the four sacred rivers. In the ancient cosmology of Egyptian Heliopolis, the first manifestation of being (Atum) from the watery abyss of non-being (Nun) was pictured in the shape of a hill or mound. This pattern also reminds us of the pyramid shape of ancient Egypt. The Pyramid is like a three-dimensional Tetractys. It is difficult for us to judge to what extent these number patterns were consciously refected in ancient sacred architecture.

The Tetractys pattern also has relevance for the study of Harmonics. One example that is often cited is Plato's LAMBDA pattern given in the Timaeus (35b-c). According to Plato, who is heavily influenced by Pythagorean arithmetical cosmology, the Divine Artificer uses this pattern in order to create the 'world soul.' He portions numbers into two sequences called the Circle of the Same (1,2,4,8), and the Circle of the Other or Different (1,3,9,27), which are arranged in the familiar pattern, shown as Figure 5. These seven numbers are then associated with the seven planetary lights. It is natural that the missing portions of the pattern are to be filled in, to complete the Tetractys shape. This was done by later writers like Crantor and Nicomachus.

The two arms of the Lambda fulfil the symbology of the Tetractys, because the numbers move from 1 (the generator), to the linear (prime) numbers 2 and 3, to the plane numbers 4 and 9, to the cubic numbers 8 and 27.

Numbers can also be examined as componants of the HARMONIC SERIES of a vibrating musical string. The open string is

the 1st Harmonic, the generator tone which is the source and potentially manifests all of the harmonics. We will use the pitch C (arbitrarily) as our reference pitch. Now the harmonics 2, 4, and 8 will also yield pitch C in higher octaves. Hence this sequence is called the Circle of the Same. Powers of the number 2 give octaves, associated with cyclical identity. The other sequence yields ever new pitches; hence it is called the Circle of the Other. In harmonics 1,3,9,27 becomes C:G:D:/A--a line of fifths. It is convenient to look at the harmonics as if they are all within one octave, since musical patterns repeat at the octave. In order to do this for our sequence, we divide the number by the appropriate power of 2, so that the ratio lies between 1/1 and 2/1. The resultant ratios give us the accurate frequency relations for that musical interval. Thus G is ratio 3/2 (Sesquialter), D is ratio 9/8 (Sesquioctave), and /A is ratio 27/16 (the Pythagorean comma-raised sixth). Hence number patterns can be converted directly into musical intervals by employing the

if C is the 16th harmonic, then D is the 18th harmonic, and so on. The relation between the 18th harmonic and the 16th harmonic is 18/16 which is 9/8, yielding pitch D. Ancient writers are consistent in the use of reciprocals. We can derive the reciprocity from the simple fact that pitch can both rise and fall.

The monochord string may be divided into any number of arithmetical divisions, generating all of the scales used in the ancient music cultures. Numbers become harmonic patterns. Certain numbers that have prominant cosmological significance also turn out to be significant numbers from the stand-point of musical harmony. The evidence leads me to conclude that the significant musical patterns have generated the cosmological numbers.

The use of the monochord in this manner is called the ARITHMETICAL PROPORTION, and is exceedingly ancient. But the monochord is not the only means by which just musical scales can be constructed. The other time-honoured procedure is called the HARMONICAL PROPORTION. By this method, musical strings are tuned to each other through the use of Harmonic Series componants. For example, if I wish to tune my G string as a perfect 3/2 ratio to my given C string, then I would form a beatless unision between the 3rd Harmonic of my C string, and the 2nd Harmonic of my G string. This tuning method pairs with the monochord method in a reciprocal manner. We shall see this same reciprocity later in the Arithmetic Mean and the Harmonic Mean, which can also be derived from the Tetractys.

Returning to our Lambda pattern, it is highly significant that the numbers involved, as well as the numbers in our monochord sequence, have a special property. They are all what modern tuning theorists call 3-LIMIT NUMBERS; that is, all can be decomposed into factors of the prime numbers 2 and 3. In other words, 3-Limit numbers have the form 2^a*3^b. All intervals inherent within our sequence are expressed as ratios using this 'family' of numbers alone. Now this family of numbers is a sub-set of the larger family of numbers that we describe as 5-LIMIT NUMBERS, which uses factors of the form $2^a*3^b*5^c$. Such numbers make up the mainstream of tuning culture throughout history among all of the ancient cultures. The ubiquity of 5-Limit numbers in ancient cosmology has led historians of mathematics to call them Regular Numbers, or Babylonian Numbers. In the west, musicologists have designated 3-Limit numbers as Pythagorean numbers.

Looking at our Lambda pattern, a music tuner would describe this harmony as a line of musical fifths. All 3-Limit number sequences produce a line of fifths. Obviously, this is the case due to the special nature of the octave (the number 2), which results in pattern identity. Out of this fact comes the principle of OCTAVE EQUIVALENCE. Only one octave is needed to explore harmonic patterns in general. As a result, we can simplify the Lambda pattern into a line of fifths, eliminating octave redundancy. This is shown in Figure 6. Note that, even in

this more abstract version of harmonic relations, a simple 'pebble pattern' is sufficient to describe tuning procedures.

Now we understand why the numbers 1 and 2 are so special to Harmonics and Arithmology that they are called PRINCIPLES OF NUMBER. The number 1 (the Monad) is present in all harmonies since every number is divisible by 1. All other numbers are a mask of the Monad. The number 2 is cyclical identity, and, as the octave, has a special relation to the Monad. It is the matrix (mother) number within whose boundaries all harmonies are born. It is the 'womb' of harmony, leading both ancients and moderns to describe the octave as a circle. The number 3 is the first number to generate new elements in the scale. Specifically it is the prime numbers that are the 'hero' numbers, that create new harmonic elements, that generate new offspring within the double. Composite numbers are but octaves of primes or combinations of primes. The number 1 is androgyne. The number 2 is female. Hence all even numbers are female, since they are divisible by 2. Odd numbers, and especially prime numbers, are male. They are the form numbers which unite with even numbers (matter) to generate actual harmony (the matrix of relations).

In the Timaeus, Plato begins with the Lambda Tetractys, but then he instructs us to convert it into a monochord scale. He expands this scale into a more complex pattern by employing MEANS. Through the Means and another procedure familiar to music tuners, he instructs us to find the division for the sacred OGDOAD of harmony. We will follow his instructions a little later in this essay; but first, it is wise to bring out yet more musical implications of the Tetractys pattern.

I mentioned above that 5-Limit scales are the mainstay of musical harmony. Just as we used the Tetractys and our line of fifths to define 3-Limit harmonic structures, in the same manner it would be useful for theoretical and practical tuning purposes to be able to notate 5-Limit musical scales. As it turns out, there is an ideal model for such a matrix, and it is--the Tetractys! If we assume the principle of Octave Equivalence, in order to eliminate octave redundancy in our matrix, then we can convert our horizontal line of fifths into a triangular pattern showing the interaction of the powers of the numbers 3 and 5. The model is shown as Figure 7. Note the similarity in conception to the matrix model in Figure 5. 3-Limit harmony was defined by the 'seed' consonances of the fifth (3/2) and fourth (4/3). 5-Limit harmony is defined by the 'seed' consonances of the major sixth (5/3) and its reciprocal, the minor third (6/5), and by the major third (5/4) and its reciprocal, the minor sixth (8/5).

These intervals together form the THREE AXES and SIX TUNING DIRECTIONS of Just Intonation tuning. Just major triads (eg. C-E-G) form a simple 'up' pointing triangle on the matrix; whereas, just minor triads (eg. C-Eb-G) form simple 'down' pointing triangles. Every ratio using the prime factors of 2, 3, and 5 can be located on this triangular grid.

The first four prime numbers, that is, 1, 2, 3, and 5, are the power numbers that generate the norms and the complexities of musical harmony. Interestingly enough, it is the roots of the same numbers that are omni-present in the constructions associated with sacred geometry. As an example, Figure 10 shows the generation of these sacred roots from the square. There is much ancient symbology and allegory around these sacred roots. Sometimes the roots are reduced to a trinity of numbers, due to the special relation of numbers 1 and 2. The ancient Hindus, Babylonians, and Egyptians envisioned a trinity of highest gods. In the Babylonian-Sumerian tradition, where musical arithmetic was especially evolved, the gods were given numerical values. Ea-Enki=40 which means 40/60=2/3, Bel-Enlil=50 which means 50/60=5/6, and Anu-An=60 which means 60/60=1/1. Thus they relate by 4:5:6--a musical triad. 60 was associated with 1 in their sexagesimal arithmetic (which, by the way, is perfectly suited to monochord calculations); consequently, these gods are linked with the powers of the prime numbers 3, 5, and 2 (and/or 1) respectively. It is interesting that, within the oldest known strata of Babylonian arithmetic, is found the ancient Table of Reciprocals. It lists all of the 5-Limit numbers between 1 and 60 (which are deified), and carries on so that the last numbers are 80:81 (an extremely important musical ratio found in just intonation harmony, called by moderns and by Philolaus the COMMA). It is clear that early mathematics was intimately entwined with musical procedures.

The sacred roots were associated in ancient times with the four 'elements,' as given by the Pythagorean philosopher Empedocles. These were also connected with the simplest patterns in 3-dimensional geometry--now called the Platonic Solids. Number 1 is FIRE, the source of transformation. Heraclitus said that all elements are an exchange for fire, which is literally true in harmonics. It is connected to the Tetrahedron, the simplest geometrical solid. (It is assumed that all of the solids are derived from and contained within the Sphere, which is the field of all possibilities). The number 2 is EARTH, the mother-goddess, the Hearth of harmony, the middle (Mese), the octave, the Cube. The number 3 is AIR, which separates and multiplies being, the number of power, the Octahedron. The number 5 is WATER, the number of beauty, which generates the 'flood' of harmonic complexity, the Icosahedron.

In examining the esoteric aspects of Pythagorean philosophy, we find reference to the fifth element, usually called the Aether, and associated with the Dodecahedron. It is the prime number 7. In harmonics, the number 7 is the 'gateway' number to a whole new order of complexity in the matrix of harmony. We found that 3-Limit harmonies can be graphed in a linear fashion; while 5-Limit harmony can be graphed as a 2-dimensional triangular matrix. We can extend this procedure (as instructed in Plato's Epinomis) so that 7-Limit harmonies can be graphed as a 3-dimensional matrix based on the tetrahedron. We may now go back to Figure 1. The point refers to 2-Limit harmony (which is but octave positions of our generator note and hence a

point on the matrix). The line gives 3-Limit harmony. The surface gives 5-Limit harmony. And the solid gives 7-Limit harmony. We have now integrated all of the prime numbers smaller than 10.

For those who doubt that 7-Limit harmony is implied by the ancient writers (and there are many moderns who doubt that the ancients were capable of comprehending this advanced level of arithmetic), there is ample evidence. I shall give only a few examples. Ptolemy, in his manual of Harmonics, gives the traditional Greek tunings and their associated champions. Among them are the 7-Limit scales attributed to the Pythagorean philosopher Archytas, who so influenced Plato. Plato himself gives many monochord divisions in his dialogues. He often disguises them as 'cities' defined by numbers, whose citizenry must relate in a harmonious manner. In the Laws we find the city of Magnesia, which veils the monochord double 2520:5040. This double happens to be one of the most significant 7-Limit monochords. Other examples can be given which show that the ancients were not ignorant (as are many moderns!) in regard to the implications of tuning theory. Indeed, the monochord was imbedded in their educational system to such an extent that the average educated Greek was expected to understand these procedures. Music and early philosophy cannot be meaningfully separated. Many of the subtle allegories implied in Plato's writings are lost to us because we do not appreciate the musical contexts of the story. He wrote these dialogues as intellectual-spiritual entertainment for an educated audience who delighted in such musical allusions.

Although this is not the format to explore it in any comprehensive manner, it can be shown that much of early philosophical speculation can be derived from the problems that monochord work generates. A prime example involves the dichotomy between the Pythagorean concepts PERAS (Limit), and APIERON (Unlimited). When setting up a scale, one must specify the elements (ratios) and sacrifice all of the other possibilities. In fact, harmonic complexity must be balanced and held in check by some numerical rubric which establishes order. We have already seen the concept of Limit at work in the restriction of factors to the powers of simple prime numbers. Yet we have also seen the need to increase the 'numerosity' of patterns through least common multiples in order to express the sequence in a practical manner. The number series is potentially infinite--it is the repository of the Unlimited. As Anaximander said, all 'being' is separated out of the potential infinite by reciprocal operations. Thus we see why the Square Number series is associated with Limit. The odd and prime numbers generate new harmonic material. They define new elements in the scale. The even numbers generate 'numerosity' or complexity through octave doublings. Thus the number 2 is associated with the Apieron. It is the power of potential complexity which issues from the initial unity (Monad). All musical tuning starts from Unity (the reference pitch) and proceeds to some level of complexity (the desired scale).

The Tetractys has still more lessons to teach us about

musical harmony. For it is implied among the ancient writers that the concept can be envisioned as a series of ratios using the numbers up to 10. The number 10 is given special significance as an important boundary point. The ratios implied by the first 10 numbers are the most powerful of musical norms, the intervals which are the seed-bed of higher musical structures. These ratios, when reduced to one octave, are: 1/1(Unison), 2/1 (Octave), 3/2(Fifth), 4/3(Fourth), 5/3(Major sixth), 5/4(Major third), 6/5(Minor third), 7/4(Augmented sixth), 7/5(Augmented fourth), 7/6(Augmented second), 8/5(Minor sixth), 8/7(Septimal wholetone), 9/5(Minor seventh), 9/7(Diminished fourth), 9/8(Major wholetone), and 10/9(Minor wholetone). Harmonies made with more complex primes (eg.11, 13, 17 etc.) are highly exotic, complex, and strange--outside the mainstream of tuning history. The last two ratios given, 9/8 and 10/9, relate by the Syntonic Comma (ratio 81/80), which is the smallest interval needed for most traditional tunings.

Our 5-Limit matrix of Figure 7 can be 'transposed' so that the generator sits in the middle position of the Tetractys, as shown in Figure 8. This position is useful since it allows the Tetractys to generate a number of the most significant historical and cosmological monochord sequences. These tunings are all subsets of the 'pebble pattern' given by the Tetractys. Again we see the pattern as the fountain of highly significant harmonies.

Figure 9 gives a very simple monochord. The string is divided into eight units; hence the Mese sits at position 4. The monochord scale yields the just minor triad associated with the Sub-harmonic Series. The Over series yields the just major triad defined by the Harmonic Series. I have presented this scale in order to show a general property of the number sequences. The Harmonic Series is ASYMMETRICAL; thus it pairs in 'yin-yang' fashion with its reciprocal.

The vast majority of harmonic patterns have this property of asymmetry. But there are a minority of patterns which exhibit SYMMETRY. In other words, they give the same scale as Harmonic Series componants, and as string divisions. Since such harmonies are comparative rarities, they were afforded great metaphysical significance in the ancient science. The simplest possible symmetrical harmony is shown on Figure 11. It is often called 'The Musical Proportion' or 'Mousikē.' Its symmetrical properties allows us to use a double-colon, as shown. In music, this harmony is called the Suspended Triad. It is also called the sacred Pythagorean Triad. It defines the Axis of the scale by displaying the fixed boundaries of the two tetrachords of Greek harmony. In modern terms, it is the Dominant and the Sub-dominant. It also defines the PERFECT CONSONANCES of music: ie. the 1/1, 2/1, 3/2, and 4/3. Note that these consonances are all 3-Limit. The 5-Limit elements in the scale were considered 'movable' or changeable, providing the IMPERFECT CONSONANCES. Thus we can associate the Perfect consonances with the first four numbers, and hence the Tetractys.

The Mousikē is not the only symmetrical harmony to have great musical and cosmological interest. Figure 12 shows the sequence 60:120. In this harmony both the Perfect Consonances and the Imperfect Consonances lie in a symmetrical sequence in smallest integers. This is the core structure of 5-Limit harmony. It also forms the core of the Tetractys pattern. This monochord, and its companion, the sequence 30:60, is closely associated with Babylonian cosmology and mathematics. At the end of the Greek era, the great Egyptian scientist Ptolemy used the double 60:120 as his reference monochord division; moreover, he expressed ratios which involved fractions in the old Babylonian way using base 60 numbers (like our division of degrees into minutes and seconds). Thus he showed his debt to the pioneers of monochord mathematics.

The double 60:120 displays a 'triple-strand' or three-layered pattern which we associate with chromatic tunings. Further expansions of this triple-strand in symmetrical arrays also form important cosmological monochords. Two examples are given in Figure 13. A wide expansion of the triple-strand tuning is at the heart of the ancient tuning system of India (described in the Nāṭyashāstra) and Babylonia. The double 360:720 is associated with the calendar (since the year consisted of 360 days, or 720 days and nights, with five days at the end of the cycle which are 'outside time'). The numbers 432 and 864 come up in various measures of cosmic time.

Figure 14 shows one of the most important asymmetrical monochord sequences. These two scales are the smallest integer expression of the two JUST DIATONIC HEPTADS, which give the diatonic scales. This is a 'double-strand' tuning characteristic of 5-Limit diatonic harmony. Note that the Tonic (C) is not in the central position of the Tetractys. We can see the same harmonic structure with the C in the middle by examining Figure 16. This is another mode of the same Diatonic Heptad. These tunings are central to both Greek and Hindu cultures. The sequence 30:60 is sometimes called the Babylonian series, for obvious reasons. In their tradition, the first 60 numbers were deified as gods. This scale is also one of the two scales that Plato 'allowed' in the Republic. The other one is shown in Figure 18, namely the monochord 432:864. The reason that he restricted the harmony in the city to these two scales seems clear. Any other mode of the diatonic can be derived from these two paradigms. In other words, nature provides us with two paradigms for the diatonic scales: a 3-Limit version which is a 'single-strand' line of fifths, and a 5-Limit version which is a 'double-strand.' As is the case in many of Plato's references to musical arithmetic, he is giving us a lesson or highlighting some aspect of harmonic structure, usually doing it through allegory for the spiritual entertainment of his audience.

The other 3-Limit sequence in Figure 18, the double 72:144, is an exceedingly ancient scale. It is the old pentatonic scale of ancient China placed in its symmetrical mode. It also turns up in India and Egypt. When this sacred double is given in

its 5-Limit sequence, (not shown here), it generates the smallest integer appearance of the double-strand Ogdoad. It also stands in a special relation to the double 30:60, since the harmonies 'exchange places' between the Over and Under sequence. Monochord divisions can always be paired in this way (and other ways). Thus the various sequences can be related to each other in subtle ways, forming 'kinship' connections. This aspect of monochord mathematics can become rather complex, and I will not pursue it far in this essay.

These two scales in Figure 18 are not subsets of the Tetractys, but I could not resist including them in this small sampling of classical monochords. The numbers have various allusions to the Bible and other literature heavily influenced by Babylonian and Egyptian culture. The fact that the same tunings appear in all of the ancient cultures indicates that monochord lore was probably an ancient common spiritual language among all of these cultures. This common spiritual lore has been lost as the monochord has become 'ghetto-ized,' being of interest mainly to only a small band of academic musicologists and tuning theorists.

Figure 17 denotes a core double-strand pattern, a pattern which is then 'expanded' to form the Diatonic Heptad. In my other writings on tuning theory, I have named double-strand tunings PTOLEMAIC TUNINGS, in honour of the great musical scientist from whom we have the earliest explicit notation of these scales. As seen above where the sequences 30:60 and 72:144 are 'cousins,' cosmological monochords tend to come in 'families.' An even closer family relationship is seen in the sequences ruled by the doubles: 6:12, 12:24, and 24:48. In the Timaeus construction, we will see another illustrious member of this family (double 96:192) honoured.

Figure 15 is included in order to show that not only can important harmonic scales be derived from the Tetractys pattern, but also important musical intervals which clarify the structure of just tuning. On the left is the microtone C#-Db, called a DIESIS or ENHARMONIC. According to Aristoxenus, the Diesis is the smallest interval to have independant status as an element in the scale. The Greeks categorized musical harmonies into three camps, called the three Genera: Diatonic, Chromatic, and Enharmonic. Diatonic harmonies are double-strand, Chromatic harmonies are triple-strand, and Enharmonic harmonies are 'quadruple-strand.' Thus the classification of 'levels of resolution' for the harmony matrix relates to the levels of the Tetractys.

The field (matrix) of harmony is amenable to various levels of resolution. Our modern approach is to divide the octave into 12 medial semitones. In such a division, the Diesis dissappears, for we use one compromise pitch to represent both C# and Db. However, it is also possible to divide the octave into a scale of Enharmonics, in which case there are 31 notes in the octave. These 31 notes are then used as a 'pool' of elements

that can be potentially selected for seven-tone scales. Using this resolution, the Enharmonic is one step in our collection of 'bricks' with which we measure the architecture of musical scales. Larger musical intervals can be expressed in the number of steps involved. In this case, the small chromatic semitone is two steps, and the large diatonic semitone is three steps, as shown in the Table.

An even more refined resolution of the octave matrix uses the Comma as the step interval. There are 53 Commas to the octave. Now the Diesis is two steps in this harmony system, the Chromatic Semitone three, and the Diatonic semitone five. In the classic Vedic book on tuning, the Nāṭyashāstra, the Comma is called a SRUTI, and is the primary building block. All intervals in the scale can be measured in sruti-steps. This approach is at the core of the ancient music cultures of India and the West.

A third level of resolution is given in the Table. Here the octave is resolved into an extremely refined scale, based on the musical SCHISMA. This interval was probably also known in ancient times, since there are oblique references to it in the fragments of Philolaus, and in Plato. I will not pursue this depth of harmony resolution in this short paper. I bring this information only to show that the field of harmony is open to many alternative scales and levels or resolution. The tuner must choose one in the here-and-now and sacrifice the others in the process. The exploration of the various possibilities leads one to a frame of mind where alternative patterns can be considered according to the local context. This way of thinking is crucial to an understanding of Vedic writings and western writings about harmony. The meaning of the numbers is always context dependant; therefore, there can be no dogmatic 'right or wrong' tuning. In the Greek tradition, it was considered an honour to discover a new one. In the west, this tradition was later destroyed when medieval culture imposed one tuning ideal on Europe for hundreds of years (until the 15th Century and the revival of intonational plurism). The 19th Century again imposed a tuning 'monotheism' on European culture, which threatens to subvert the rich complexity of world tuning culture.

Figure 20 gives yet another mode of the Just Diatonic Heptad. It is ruled by the double 36:72. Note the extra note in the scale, defined by number 50. Normally this note would be omitted. I have included it here to complete the set of all 5-Limit numbers between 36 and 72. We could also have inserted 50 in the sequence ruled by 30:60. In the sequence 24:48, we could have inserted 25 in order to complete the sequence. Of course, it is not necessary or even desirable to do so. But it shows that we can generate various diatonic scales by setting up sequences of 5-Limit numbers. Of course, if we set up the complete sequence between 360:720, there will be a lot more elements in the scale (it will be chromatic or even enharmonic instead of diatonic). The use of large numbers integrates more potential elements into the scale. We always have the choice whether or not to include these elements in the scale.

Speaking of 50, remember our Babylonian god Bel? He was an expression of the power of the prime number 5 in harmonies (as is Osiris in the Egyptian tradition). In the three diatonic scales given, he appears as the 'playful' aspect or the 'exhuberant' one in the numbers 50 and 25. In Sumerian mythology, he is the god who sits 'on top of the mountain.' This quality is typical of 5 and its powers. We see this tendency even in the simple monochord sequence 3:6 given on Figure 21. Without the 5, we would have a simple 3-Limit sequence 3:4:6. The number 5 adds an expansive element. We see the same situation in the sequence 4:8 of Figure 9. Figure 22 is a special case, since the number 5 is here placed in a dominant position in the sequence. The number 10 is 2*5. The numbers 10 and 5 were highly revered by the ancient tuners. Note that many of the most significant monochords relate by 10; for example, 6:12 and 60:120, or again, 432:864 and 4320:8640. The numbers 10 and 5 create the intervals of beauty in music, but they also proliferate 'numerosity' and interval complexity. Many of the 'secrets' of ancient writings on harmonics focus on relations between 3-Limit and 5-Limit harmonies.

The monochord 3:6 is the simplest calibration of the just major and minor triad, which are the core structures of 5-Limit harmony. This sequence is also a member of the same family as 6:12 and 24:48. Note, on Figure 11, that the 3-Limit symmetrical Mousikē sequence can be converted into an asymmetrical 5-Limit sequence by inserting the number 10. Thus we would have 6:8:9:10:12, as harmonics C-F-G-A-C. Even though the 'playful god' is not included in the Mousikē sequence, he is potentially there as an integral element. Such is the nature of harmony!

We are almost ready to complete Plato's Timaeus monochord, but first one must be clear about the concept of MEANS. This framework, first described by Archytas, is of fundamental importance in understanding the metaphysical foundations for our monochord explorations. It also justifies the sexual imagery associated with numbers. The Greeks, Egyptians, Babylonians, and Chinese saw life as a balanced interplay of polarity or a 'middle path' between extremes.

Through the concept of Means, orderly number relations are generated between the opposites, as 'offspring' are generated from the two 'parents.' We shall see that the 'seed' archetypes of 3-Limit, 5-Limit, and 7-Limit harmony can be thus 'birthed.'

There are three traditional Means, Harmonic, Arithmetic, and Geometric. Eudoxus added the three compliments to give six, then later writers increased the types to 10, in honour of the Tetractys. In this paper we shall only consider the original three, since these three will suffice to generate all of harmonic 'space.' Of these three, the Geometric Mean (G.M.) is special and esoteric. We shall save it to last.

The Arithmetic Mean (A.M.) and Harmonic Mean (H.M.) can be generated by simple formulas. If a and c are the two extremes, and b is the mean, then, for the A.M. b=a+c/2, and for the H.M. b=2ac/a+c. We can see the simplest examples of these two means by examining the Tetractys once again. In the Lambda of Figure 5 we see the A.M. expressed in the sequences 2:3:4, in 4:6:8, and in 6:9:12. We also see the H.M. in the sequences 3:4:6, in 6:8:12, and in 9:12:18. The Tetractys offers the most abstract expression of these number relations.

However, this approach is still somewhat too abstract for our musical application. We wish to look at the means in relation to ratios. Fortunately, there is a simple rule which makes the generation of these two means quite easy. In order to find the two means between a given ratio, simply double the numbers and insert the means. For example, we want to find the means of the octave (2/1). Double the numbers to give 4/2, and insert the mean term. 4/3 is the H.M., and 3/2 is the A.M. Figure 19 gives the Means for the most important harmonic intervals. Let us examine this Table more closely, since it contains much information relevant to music and metaphysics.

First of all, we see that the G.M. of an interval ratio is always an irrational ratio (ie. it cannot be expressed in the form a/b). This fact has profound implications for tuning theory and monochord work, to be examined shortly. Note however, that in the abstract form of the Tetractys given in Figure 5, there appears a 'continuous geometrical proportion' between the numbers of the two arms. Thus the G.M. is present within the Tetractys.

The H.M. and A.M. generate the 'seed' ratios and consonances of harmony. The Octave generates the Fourth and Fifth, the frames of the tetrachord and the Mousikē monochord of Figure 11. These are the founding ratios of 3-Limit harmony and the line of fifths. Plato, in the Epinomis, instructs us to generate the other means in turn. He gives the successive duplications of Unity as 1:2, 2:4, and 4:8 which he calls "the spacial and the tangible." One cannot help but see in this procedure the essence of the monochord 'game.' Starting from the open string (1), we first establish the Mese through the sequence 1:2. The next possible division gives the sequence 2:3:4, establishing the 3-Limit harmonies. The next possible sequence is 3:4:5:6, which yields the 'seed' 5-Limit ratios of our triangular matrix. Finally, the next possible sequence is 4:5:6:7:8, which generates the seed ratios of 7-Limit harmony and the tetrahedral matrix. Again we are reminded of the four levels of the Tetractys in the levels of ratio interaction.

In what is probably the most beautiful description of this monochord metaphysics in Plato, he says in the Epinomis: "In fact, what is divine and marvelous for those who understand it and reflect upon it is this--that through that power which is constantly whirling about the duplication, and through its opposite, according to the different proportions (ie. through the different kinds of means), the pattern and type of all nature

Proportion with just one more ratio, the cube root of two. In ancient times, this problem was called "the duplication of the cube" or the establishment of two mean proportionals. The earliest and most brilliant known solution is by Archytas. Thus it was within their capability to erect this equal-tempered scale.

Just as the $\sqrt{2}$ is associated with the Diatonic Genus (through the 9/8 ratio between F and G) and 12-E.T., the ratio $\sqrt{6}/2$ (the neutral third) sits midway between 6/5 and 5/4. These two just ratios relate by the small Chromatic Semitone (25/24), which is associated with the temperament 19-E.T. This tuning system uses chromatic semitones as its step interval, nineteen to the octave. We can associate it with the Chromatic Genus. The G.M. of the fourth yields $2/\sqrt{3}$, which is the signature interval of 53-E.T., the scale of commas or srutis. In other words, if we want to calibrate 53-E.T. on the monochord, we must use $2/\sqrt{3}$ and $\sqrt{3}$ as our 'gnomon.' Last but not least, the major third yields the G.M. $\sqrt{5}/2$, a remarkable interval mid-way in size between the wholeteps 9/8 and 10/9; hence it is called the MEANTONE. This interval is closely associated with the great 31-E.T. tempered system, and with the whole family of ratios related to the Golden Section. Thus it has much significance for harmony as well as geometry.

These systems of musical temperament are the BEST POSSIBLE EQUAL-TEMPERED DIVISIONS of the octave. It is intellectually satisfying that they can be generated by the three sacred roots of geometry. Looking at Table 19 as a whole, we see the same four levels of 'beingness' that were implied by the abstract Tetractys pattern of Figure 1. Also, it is tempting to relate the four tempered divisions to the four 'elements,' with the master cyclical temperament (612-E.T.) being the fifth, esoteric element. Or again, we could relate them to the five Platonic solids, transformed to a musical context.

At last we can return to Plato's 'World Soul' monochord in the Timaeus. Here we leave the complex and rarified atmosphere of irrational ratios and return to traditional monochord methods. As usual, Plato is weaving a musical lesson in his story, a lesson involving three highly significant harmonies: the Pythagorean Tetrad, The Pythagorean Ogdoad, and the Just Ogdoad.

As we saw earlier, he begins with the Lambda of Figure 5. The harmonies of the two series are Unity (the Same) and the Pythagorean or 3-Limit Tetrad (the Other). In naming these series 'circles' he is alerting us to the monochord context; for it is common for musical scales to be described as circles (since the pattern repeats at the octave). Now the Pythagorean Tetrad is a highly significant harmony, since it is the widest expanse of the line of fifths that one can encounter before one crosses the 'comma boundary' and encounters comma-altered dissonances. We see this Tetrad expressed within the comma boundary as F-C-G-D in the double 24:48 (Figure 14). In the initial construction, the mode given by Plato is C-G-D-/A.

Note that I am only looking at the Over Series here, as in many of the sequences in this paper, in order to save space. The reciprocal, however, is always implied. There are also other significant relations between the Over and Under series which I cannot explore in this essay.

Now Plato requests us to insert the H.M. and A.M. between the members of the two series, as shown in Figure 24. In order to do this, multiply the series by six, so that 1:2:4:8 becomes 6:12:24:48. When this is done to the two series we get the Circle of the Same (the unchanging tetrachord frames), and the Circle of the Changing (the 3-Limit Tetrad).

Next he requests that we inspect this material for the presence of musical fourths between the members. In order to do this, we must convert these two series into a single monochord sequence, using a least common multiple to make all elements 'communicable and commensurable.' This is accomplished in smallest integers by multiplying by sixteen, yielding the monochord sequence ruled by the double 96:192. An inspection of the Over series reveals fourths between C-F, D-G, and G-C. Now he wants us to 'fill up' these fourths by using the 3-Limit tetrachord 192:216:243:256. In order to do so, we must multiply our sequence by sixteen again, finding the smallest integer expression for our 3-Limit Ogdoad: the double 1536:3072. At this point Plato tells us that we cannot continue because "by this time the mixture from which he was cutting off these portions was all used up."

That last sentence is a clue to the musical lesson hidden in this construction. For the Pythagorean Ogdoad is the furthest one can expand the line of fifths before one encounters quasi-just elements which are sonically indistinguishable from 5-Limit intervals. The clearest example of this reality can be seen by examining the expanded field of just harmony given on Figure 27. If we begin at C and move to the left, the ninth member of the series is an E which is only a schisma flat of the just 5-Limit E. In other words, any expansion beyond eight pitches in the 3-Limit line of fifths results in schisma-altered versions of 5-Limit intervals. Hence we see the creation of the Schisma Boundary in the matrix. Note this 'four beat rhythm' in the line of fifths. Indeed, three of the most sigificant 3-Limit harmonic structures (considered sacred by the ancient scientists) are the Pythagorean Tetrad, the Pythagorean Ogdoad, and the Pythagorean Dodecad (or 3-Limit 12 note scale). I shall not consider the Dodecad in this paper, but only note the progression from four to eight to twelve in the line of fifths.

As if this was not enough of a musical lesson, there is another which is even more remarkable! For this construction implies the sacred 5-Limit Ogdoad shown at the bottom of Figure 24. Here are the reasons. The construction directly gave us the 3-Limit Ogdoad, which is the furthest expanse of the Diatonic scale before a major boundary is crossed. As a line of fifths we notate it F-C-G-D-/A-/E-/B-/F#. But there is a highly significant

harmony which is a 'close cousin' to this one. We notate it as F-C-G-D-A-E-B-F#. It is the double-strand or 5-Limit version of the same scale. These two scales constitute the TWO PARADIGMS OF DIATONIC HARMONY. In order to express (in smallest integers) the 5-Limit version as an Over series like our 3-Limit paradigm above, we must use the double 96:192--the same double Plato implied above in the 3-Limit construction. Thus he is 'hinting at' the existence of this important harmony, even though he does not give it directly.

The 5-Limit Ogdoad also has special properties that remind us of its 3-Limit cousin. It is also 'full;' that is, it is complete within its boundaries. Any expansion of the double-strand beyond these eight yields comma-altered versions of the scale. For example, suppose we add \D on the left. Now the scale has \D and D. Or again, suppose we add /A on the right. We now have A and /A. Hence in this harmony, like its cousin, the material is 'all used up.' An inspection of the field of harmony on Figure 27 reveals just how significant this Ogdoad is within the structure of the harmony matrix.

One of the biggest lessons to be learned from Plato's construction is this: the more we know about the deep structure of musical harmony, especially as it relates to monochord procedures, the more sense that can be made of much ancient number lore, which often seems baffling or incomprehensible without the musical key. My work in harmonics over the last 20 years has given me an immense appreciation for the advanced level of awareness afforded to musical structure by the ancient writers, who viewed the whole cosmos as obeying musical laws. Indeed, Plato goes on to use this Timaeus construction as a model for the structure of the solar system. In hindsight, we can appreciate why this is so. It was difficult for the ancients to measure the distances between the planets, but the mathematics of musical tuning is easily quantified and verified on the monochord. Thus it became the empirical base and springboard for much cosmological and metaphysical speculation.

I wish to return once more to the sacred Tetractys, in order to see it as the source for yet another great musical and mathematical model--the multiplication table and the Lambdoma. The 10*10 multiplication table is exceedingly ancient. In Italy it was traditionally called the PYTHAGOREAN TABLE. There is good reason for this association. On Figure 25 we see a fragment of the multiplication table, as far as 6*6. Even though it is most conveniently a square grid, it can be expressed as a diamond or masculine square in order to emphasize the 'emanational' nature of the table. Note that the Square Number series occupies the vertical and divides the two halves of the table.

Since the study of harmonics is, in the abstract, the study of number relations as ratios, the whole field of possible ratios can be beautifully expressed as derived directly from the multiplication table. In the Lambdoma, the Over Series which defines harmonics, is expressed as the sequence on the right:

1/1, 2/1, 3/1, 4/1, and so on. The Under Series which define string division and the Sub-harmonic Series, is expressed on the left as 1/1, 1/2, 1/3, 1/4, and so on. The central line which separates the two halves becomes 1/1, since 2/2=1/1, 3/3=1/1, etc.. All of the ratios to be found from the Harmonic Series and from string division are thus presented in one systematic and elegant model. The overall form of this model reminds us of the Tetractys.

Note that when we connect ratios of the same size (called Identity Rays by Albert von Thimus, who rediscovered the Lambdoma), they converge at a point beyond the Monad, implying the ratio 0/0 or Silence (the Egyptian Nun). Very much metaphysical and mathematical lore can be derived from this diagram.

Indeed, it is amazing just how much of the basis of harmonics and other ancient number lore can be derived from the Tetractys pattern. The whole matrix of musical harmony is best modelled using the triangular grid for the three axes of just tuning. When we look at a wide expanse of the matrix, with the natural boundaries marked in, as in Figure 27, we are struck by the appearance of Tetractys-like patterns in the 'regional' boundary markers.

In fact, the further that I explore the deep structure of musical harmony, the more relevant Tetractys thinking has become for me. It has forced me to come to the conclusion that the ancients actually knew quite a bit about tuning theory. Modern academic attitudes tend to imply that the monochord 6:12 was the ONLY monochord known by the ancients, and that the musical fifth (3/2 ratio) was discovered only in the 5th Century B.C. But in fact musicians had been tuning harps and zithers for at least two millenia of highly musical and mathematical cultures in Egypt, Babylonia, and probably Indus. It is most unlikely that the connections between ratios and harmonies were not realized and explored. In fact, it is highly likely that the 'monochord experiment' could even pre-date urban civilization; certainly it is among the earliest scientific experiments ever investigated by humanity. Indeed, the further one traces the early history of mathematics, the more musical is its orientation.

It would be fruitful for an understanding of our common cultural-spiritual history to investigate the relations between harmonic lore, mythology, and early science. I am presently working on such a synthesis in relation to pre-socratic Greek philosophy (ending with Plato). Some day I hope to apply the same investigation to Babylonian mythology, Egyptian mythology, Vedic mythology, and Chinese mythology. All of these traditions have a common thread in monochord lore, musicalizing the cosmos in various ways. The Pythagorean lore, especially as preserved in Plato, is the key to understanding the mind-set of the ancient tuner-scientists. The image of the Tetractys is a great tool for exercising this exploration, for setting up strong associations with the actual structure of harmony.